Expansions: A Loosely Tied Traverse from Feynman Diagrams to Quantum Algebra

Geometric, Algebraic, and Topological Methods
for Quantum Field Theory,
Villa de Leyva, Colombia, July 2011

Quick Summary of Lectures by Dror Bar-Natan

Home:=`http://www.math.toronto.edu/~drorbn/`
Talks:=Home/Talks/
VdL:=Talks/Colombia-1107/

Abstract. This is a summary of a series of 6 lectures I gave in Villa de Leyva, Colombia, in July 2011. The common thread for the series were "expansions" — Taylor expansions, in some sense, yet expanded so much as to be barely recognizable. So we started from quantum field theory, where the Taylor expansion become the theory of Feynman diagrams, and continued to knot theory where expansions make sense in the abstract, and relate to some Lie theory and "high" algebra.

All lectures were videotaped and were accompanied by a series of handouts (see VdL/). Hence I will limit myself here to a quick summary and a list of links and references.

Version of May 15, 2012

Lecture 1 — The Stonhenge story. The video of this lecture is at `VdL/Video1.html`, and the handout, originally made for a lecture given in Oporto in July 2004, is on page 7 here.

The Gauss linking number of a two component link can be viewed as counting (with signs) the possible placements of a "chopstick" on the link, so that one of its ends is on one component and the other is on the other, and so that the chopstick is pointing at some pre-specified point in "heavens", which is just the S^2 that surrounds us in all directions. Inspired by this, we've set to look for and count all such "cosmic coincidences", in which one places a graph made of chopstick atop a given embedded knot K, so that the chopsticks have their ends either on the knot or at vertices in $\mathbb{R}^3$ in which several chopsticks meet, and so that each chopstick is pointing at a "star" — where a generic configuration of a large number of stars, namely points in heavens or S^2, is chosen in advance.

We determined that for the above count to be generically finite (namely for the number of equations to be equal to the number of unknowns), the "cosmic coincidence" graph we are studying should be trivalent, and so we formed the generating function $Z'(K)$ of all such cosmic coincidence counts — namely, $Z'(K)$ is a formal linear combination of trivalent graphs, where each graph D appears with a coefficient equal (up to normalization factors) to the (signed) number of times D can be placed atop K following the rules above.

We then studied in detail how $Z'(K)$ changes under deformations of K. We found that it is *not* invariant, yet when it changes, the coefficients of some triples of graphs (denoted either I, H, and X or S, T, and U, depending or the precise circumstances) jump simultaneously. Thus if we let $\mathcal{A}$ denote the target space of Z' (linear combinations of appropriate trivalent graphs) divided by the relations $I = H - X$ and $S = T - U$ (called IHX and STU), then within $\mathcal{A}$, the generating function Z' (or more precisely, a further-renormalized version Z) is a knot invariant.

At the end we noted that Z actually arises from quantum field theory. One may study the so-called Chern-Simons-Witten quantum field theory using "perturbation theory" (which we discussed in the following lecture). The resulting "partition function" is a knot invariant presented in terms of complicated "Feynman diagrams", which in themselves are complicated integrals. These integrals can be re-interpreted as "configuration space integrals", which in themselves can be re-interpreted as counting our "cosmic coincidences".

Unfortunately, the story I presented in this lecture was never written up in quite the same language. Part of the reason is that writing it precisely is harder than talking about it and allowing some impreciseness to creep in. In my mind, the best written report on the subject is Dylan Thurston's old Harvard senior thesis [Th] (where "chopstick towers" are called "tinkertoy diagrams"). The only other sources are my talk here and the video of my March 2011 talk in Tennessee, at `Talks/Tennessee-1103/`.

Lecture 2 — Perturbation theory in finite dimensions and in the Chern-Simons case. The video of this lecture is at `VdL/Video2.html`, and the handouts are on pages 7 and 8 here.

People like knots! Some evidence is on page 6 here, and we started the lecture by displaying a number of bank logos that contain knots in them, and then by reviewing Lecture 1.

We then moved on to learn about Feynman diagrams from the path integral perspective. We started with the evaluation of perturbed Gaussian integrals on $\mathbb{R}^n$, and computed these, as power series in the perturbation parameter, using what is to be called Feynman diagrams. The lovely thing is that the evaluation of Feynman diagram depends on the dimension n only very mildly, and so it makes sense to formally substitute $n = \infty$ and compute infinite-dimensional perturbed Gaussian integrals, also known as path integrals.

When all is done carefully, the result is that a path integral can be computed (more precisely, expanded in terms of the perturbative parameter) in terms of Feynman diagrams, where each Feynman diagram represents a certain finite dimensional integral whose integrand has algebraic factors coming from the "perturbative part" and Green-function factors coming from inverting the "quadratic part".

In the case of gauge theory in general, and Chern-Simons-Witten theory in particular, the quadratic part cannot be invertible due to the gauge symmetry. Yet there is a procedure called "gauge fixing", or the "Faddeev-Popov method", to resolve this difficulty. It involves integrating only over a section of the gauge orbits, and inserting a certain determinant to fix a measure-theoretic error that arises. That determinant has its own perturbation theory, and over all the resulting Feynman diagram prescription is much the same, only a bit more complicated in the details. We ended with the complete Feynman rules for the evaluation of the Chern-Simons-Witten path integral.

Much of the material in this lecture is completely standard, and can be found in any of many textbooks on quantum field theory. The best sources that are specific to the Chern-Simons-Witten theory are probably my thesis and paper [BN1, BN2] and Polyak's [Po].

Lecture 3 — Finite type invariants, chord and Jacobi diagrams and "expansions". The video of this lecture is at `VdL/Video3.html`, and the handout is on page 9 here.

Lecture 3 was a pretty standard introduction to knots, knot invariants, and finite type invariants, pretty much following [BN3].

In short, a "finite type invariant" is in a reasonable sense a polynomial on the space of all knots — that is, it is a numerical invariant which is a polynomial as a function of the knot; it is not an invariant of knots with values in polynomials, like the Conway or the Jones polynomals. Yet we have shown that every coefficient of the Conway polynomial (and likewise for Jones), in itself being a numerical invariant, is a finite type invariant.

A reasonable approach to the study of polynomials is by studying their top derivatives. The "top derivative" of a finite type invariant turns out to be a linear functional on the space $\mathcal{A}$ of Lecture 1, and hence in Lecture 3 we have posed the problem that was solved in Lecture 1 — the construction of a "universal finite type invariant".

Lecture 4 — Low and high algebra and knotted trivalent graphs. The video of this lecture is at `VdL/Video4.html`, and the handouts are on pages 10 and 11 here.

This is where the true depth of our topic begins to emerge. We first observe "low algebra" — the diagrams that make up $\mathcal{A}$ turn out to represent formulas that can be written in any appropriate Lie algebra, and hence $\mathcal{A}$ is in some sense a universal space that describes the universal enveloping algebras of *all* Lie algebras. Further, when $\mathcal{K}$, knots, is replaced with

$\mathcal{K}(\uparrow_n)$, tangles, the corresponding associated graded space $\mathcal{A}$ gets replaced by $\mathcal{A}(\uparrow_n)$, which describes also tensor powers of universal enveloping algebras. Finally, the construction of a homomorphic universal finite type invariant, or a "homomorphic expansion" $Z\colon \mathcal{K}(\uparrow_n) \to \mathcal{A}(\uparrow_n)$, becomes a matter of solving certain systems of equations universally, for all Lie algebras, a task that we name "high algebra".

We then tasted one realization of the above plan, where tangles get replaced by knotted trivalent graphs. The resulting "high algebra" that thus arises is the Drinfel'd theory of asociators. The more complete discussion of knotted trivalent graphs and associators was postponed to the following lecture.

"Low algebra" is described already at [BN3]. The story of "high algebra" in general is told at my first wClips talk at `VdL/wClips1.html` and is written in [BD2], while the relationship between knotted trivalent graphs and Drinfel'd associators is best described at [BD1].

Lecture 5 — Drinfel'd associators and knotted trivalent graphs. The video of this lecture is at `VdL/Video5.html`, and the handouts are on pages 12 and 11 (again) here.

Largely this was a review lecture, and a completion of the discussion of knotted trivalent graphs, their generators and relations, and of Drinfel'd associators. The written reference remains [BD1].

Lecture 6 — w-Knotted objects, co-commutative Lie bi-algebras, and convolusions. The video of this lecture is at `VdL/Video6.html`, and the handouts, originally made for a talk given in Bonn in August 2009, are on pages 13 and 14 here.

We started with a very brief discussion of the "bigger bigger picture". It is hardly in writing anywhere, yet see my 2010 series of talks in Montpellier at `Talks/Montpellier-1006/`, my talk in SwissKnots 2011 at `Talks/SwissKnots-1105/`, and my talk at `VdL/wClips1.html` (all are on video with links at said pages).

We then moved on to the lovely story of w-knotted objects. w-Knots are so called "ribbon 2-knots in $\mathbb{R}^4$"; locally they can be viewed as movies of flying rings in $\mathbb{R}^3$, and such flying rings may trade places either externally, as ordinary braids, or internally, by flying through each other. Thus the theories of w-braids, and likewise w-knots and other knotted objects, are richer than their corresponding "usual" counterparts (though often this richness is not seen by finite type invariants — see `Talks/Chicago-1009/`).

In parrallel with the usual "u" story, the "w" story also has finite type invariants, combinatorics (with "arrow diagrams" replacing "chord diagrams"), low algebra (related to finite dimensional Lie algebras and their duals, or equivalently, to co-commutative Lie bialgebras), and high algebra. The high algebra for the w-story arises when one attempts to construct a homomorphic expansion for w-knotted graphs, and the equations that arise are equivalent to the Kashiwara-Vergne [KV] equations that imply a relationship between convolutions on any Lie group and convolutions on the corresponding Lie algebra.

It is worth noting that there is a map from the u-world to the w-world, and this map "explains" the relationship between the Kashiwara-Vergne equations and Drinfel'd associators discovered by Alekseev-Torossian [AT] and elucidated by Alekseev-Enriquez-Torossian [AET].

We expect there to be an even higher "v-story", whose low algebra is about general Lie bialgebras and whose high algebra is the Etingof-Kazhdan theory of quantization of Lie bialgebras, but this story is yet to unravel.

The best source for the w-story is the still-evolving document and series of video clips [BD2]. Some parts that the said source have not reached yet are in my Montpellier talks (link above) and in my Bonn talk `Talks/Bonn-0908/`. The v-story is hardly written, and the most there is about it is in my SwissKnots talk linked above.

References

[AT] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators,* arXiv:0802.4300.

[AET] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld associators, braid groups and explicit solutions of the KashiwaraVergne equations,* Pub. Math. IHES **112-1** (2010) 143-189, arXiv:0903.4067.

[BN1] D. Bar-Natan, *Perturbative aspects of the Chern-Simons topological quantum field theory,* Ph.D. thesis, Princeton Univ., June 1991, `Home/LOP.html#thesis`.

[BN2] D. Bar-Natan, *Perturbative Chern-Simons Theory,* Jour. of Knot Theory and its Ramifications **4-4** (1995) 503–548, `Home/LOP.html#pcs`.

[BN3] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472, `Home/papers/OnVassiliev/`.

[BD1] D. Bar-Natan and Z. Dancso, *Homomorphic Expansions for Knotted Trivalent Graphs,* arXiv:1103.1896.

[BD2] D. Bar-Natan and Z. Dancso, *Finite Type Invariants of W-Knotted Objects: From Alexander to Kashiwara and Vergne,* `Home/papers/WKO/`.

[KV] M. Kashiwara and M. Vergne, *The Campbell-Hausdorff Formula and Invariant Hyperfunctions,* Invent. Math. **47** (1978) 249–272.

[Po] M. Polyak, *Feynman diagrams for pedestrians and mathematicians,* Proc. Symp. Pure Math. **73** (2005), 15–42, arXiv:math.GT/0406251.

[Th] D. Thurston, *Integral expressions for the Vassiliev knot invariants,* Harvard University senior thesis, April 1995, arXiv:math.QA/9901110.

A Borromean link seen at Villa de Leyva, July 2011.

Stonehenge

From Stonehenge to Witten Skipping all the Details

Oporto Meeting on Geometry, Topology and Physics, July 2004

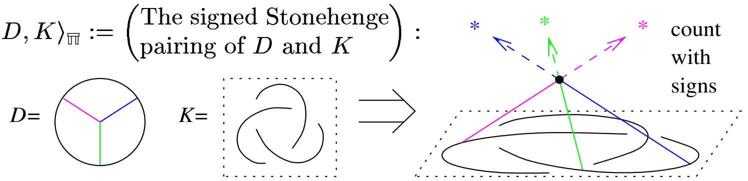
Dror Bar–Natan, University of Toronto

Witten

It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

$$\langle D,K\rangle_{\sqcap\!\sqcap} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix} :$$

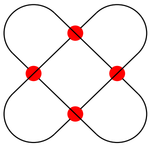


count with signs

The Gaussian linking number

$$lk(\bigcirc\!\!\bigcirc) = \frac{1}{2}\sum_{\text{vertical chopsticks}} (\text{signs})$$

Carl Friedrich Gauss



lk=2

Thus we consider the generating function of all stellar coincidences:

Dylan Thurston

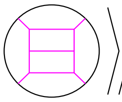
$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D,K\rangle_{\sqcap\!\sqcap} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

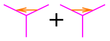
N := # of stars

c := # of chopsticks

e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle \right\rangle$ / oriented vertices

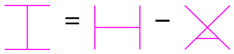


AS:  =0

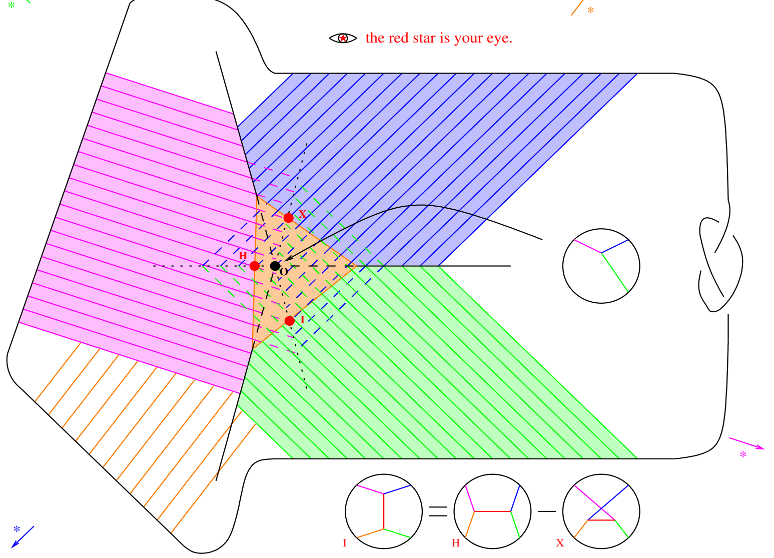
& more relations

Theorem. Modulo Relations, *Z(K)* is a knot invariant!

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,	An intersection line cuts through the knot – Solution: Impose STU,	The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.
		
(see below)	(similar argument)	(not shown here)

The IHX Relation



It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\to \sum_{D:\ \text{Feynman diagram}} W_{\mathfrak{g}}(D) \not{\sum} \mathcal{E}(D) \to \sum_{D:\ \text{Feynman diagram}} D \not{\sum} \mathcal{E}(D)$$

Shiing–shen Chern

James H Simons

V: Vector space

dV: Lebesgue's measure on V.

Q: A quadratic form on V; $Q(V)=\langle LV,V\rangle$ where $L:V\to V^*$ is linear

Compute $I=\int_V dv\, e^{\frac12 Q+P}$

$$“=” \sum_{m=0}^{\infty}\frac{1}{m!}\int_V dv\, P^m e^{Q/2}$$

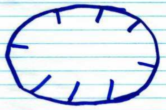
$$\sim \sum_{m=0}^{\infty}\frac{1}{m!}P^m(\partial_\varphi)e^{-\frac12 Q^{-1}(\varphi)}\Big|_{\varphi=0}$$

$\pi=2=i=1$

$$=\sum_{m,n=0}^{\infty}\frac{(-1)^n}{2^n m!n!}P^m(\partial)(Q^{-1})^n\Big|_{\varphi=0}$$

In our case,

- Q is d, so Q^{-1} is an integral operator.
- P is $\frac23 A\wedge A\wedge A$
- H is the holonomy, itself a sum of integrals along the knot K,



& when the dust settles, we get $Z(K)$!

The Fourier Transform:

$(F:V\to\mathbb{C})\Rightarrow(\tilde F:V^*\to\mathbb{C})$

via $\tilde F(\varphi)=\int_V F(v)e^{-i\langle\varphi,v\rangle}dv$.

Simple Facts:

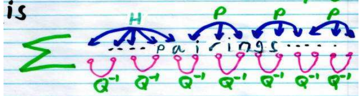
1. $\tilde F(0)=\int_V F(v)dv$.
2. $\frac{\partial}{\partial\varphi_i}\tilde F \sim \widetilde{v^i F}$.
3. $\widetilde{(e^{Q/2})}\sim e^{-Q^{-1}/2}$ where $Q^{-1}(\varphi):=\langle\varphi,L^{-1}\varphi\rangle$

(that's the heart of the Fourier Inversion Formula).

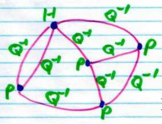
So $\int_V H(v)e^{\frac12 Q+P}dv$

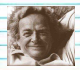
$$\sim H(\partial)e^{P(\partial)}e^{-Q^{-1}(\varphi)/2}\Big|_{\varphi=0}$$

is



$$=\sum_{\text{Diagrams}} C(D)\begin{pmatrix}\text{Products of}\\ Q^{-1}\text{'s, } P\text{'s}\\ \text{and one } H\end{pmatrix}$$

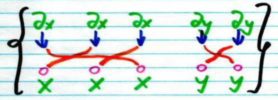




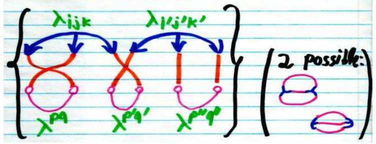
Richard Feynman

Differentiation and Pairings:

$\partial_x^3\partial_y^2 x^3y^2=3!\cdot2!$; indeed,



$(\lambda_{ijk}\partial_i\partial_j\partial_k)^2(\lambda^{pq}\varphi_p\varphi_q)^3$ is



"God created the knots,
all else in topology
is the work of man."

Leopold Kronecker

(modified)

This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

More at `Talks/Oporto-0407/`

Lecture 2 Handout

More on Chern-Simons Theory and Feynman Diagrams

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

After $A \mapsto A/\sqrt{k}$, and setting $\hbar = \frac{1}{\sqrt{k}}$:

$$Z(\gamma) = \int_{A\in\Omega^1(\mathbb{R}^3,\mathfrak{g})} \mathcal{D}A \, tr_R\, hol_\gamma(A)\, e^{\frac{i}{4\pi}\underbrace{\int_{\mathbb{R}^3} tr\left(A\wedge dA + \frac{2\hbar}{3}A\wedge A\wedge A\right)}_{CS(A)}}$$

where $tr_R\, hol_\gamma(A) = tr_R\left(1+\hbar\int ds\, A(\dot\gamma(s)) + \hbar^2\int_{s_1<s_2} A(\dot\gamma(s_1))A(\dot\gamma(s_2)) + \ldots\right.$

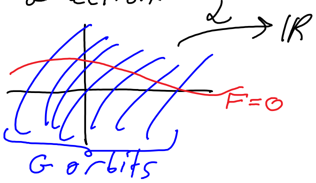
Trouble! "d" is not invertible!

Gauge Invariance: $CS(A)$ is invariant under $A \mapsto A+\delta A$, $\delta A = -(dc+\hbar[A,c])$, $c\in\Omega^0(\mathbb{R},\mathfrak{g})$

Back to the drawing board....

Suppose $\mathcal{L}(x)$ on $\mathbb{R}^n$ is invariant under a k-dimensional Group G w/ Lie algebra $\mathfrak{g}=\langle g_a\rangle$, and suppose $F:\mathbb{R}^n\to\mathbb{R}^k$ is such that $F=0$ is a section of the G-action:

$$G \to \mathbb{R}^n \xrightarrow{F} \mathbb{R}^k$$



Then

$$\int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}} \sim \int_{\mathbb{R}^n} dx\, e^{i\mathcal{L}}\delta(F(x))\cdot\det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

$$\sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^l} d\phi\; e^{i(\mathcal{L}+F(x)\cdot\phi)}\det\left(\frac{\partial F^a}{\partial g_b}\right)(x)$$

Perturbation theory for determinants?

$$\det(J_0+\hbar J_1(x)) = \det(J_0)\sum_m \hbar^m\, Tr(\Lambda^m J_0^{-1})\cdot(\Lambda^m J_1(x))$$

Berezin / Fermionic / Anti-commuting Variables: $\int d^k\bar{c}\, d^k c\, e^{i\bar{c}_a J^a_b c^b} \sim \det(J)$

So

$$Z \sim \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^k} d\phi \int d^k\bar{c} \int d^k c\; e^{i\mathcal{L}_{tot}} \text{ where}$$

$$\mathcal{L}_{tot} = \underbrace{\mathcal{L}(x)}_{\text{the original}} + \underbrace{F(x)\cdot\phi}_{\text{gauge-fixing}} + \underbrace{\bar{c}_a\left(\frac{\partial F^a}{\partial g_b}\right)c^b}_{\text{"ghosts"}}$$

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

www.katlas.org

In Chern-Simons, w/ $F(A) := d^*A = \partial_i A^i$, get

$$\mathcal{L}_{tot} = \frac{k}{4\pi}\int_{\mathbb{R}^3} tr\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A + \phi\,\partial_i A^i + \bar{c}\,\partial_i(\partial^i + ad\, A^i)c\right.$$

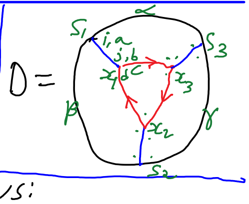
So we have

- ✱ A bosonic quadratic term involving $\binom{A}{\phi}$.
- ✱ A fermionic quadratic term involving $\bar{c}, c$.
- ✱ A cubic interaction of 3 A's.
- ✱ A cubic $A\bar{c}c$ vertex.
- ✱ Funny A and γ "holonomy" vertices along γ.

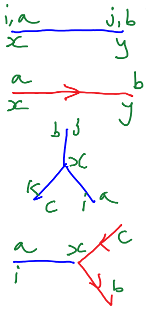
After much crunching:

$$Z(\gamma) = \sum_{m=0}^{\infty}\hbar^m \sum_{\text{Feynman diags } D} \oint \mathcal{E}(D)$$

$D =$



where $\mathcal{E}(D)$ is constructed as follows:

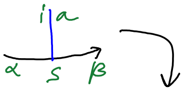


$$\longrightarrow \epsilon_{ijk}\, t^{ab}\frac{i(x-y)^k}{2|x-y|^3}$$

$$\longrightarrow \frac{t^{ab}}{4\pi|x-y|}$$

$$\longrightarrow \frac{i}{2\pi}\int_{\mathbb{R}^3} t_{abc}\,\epsilon^{ijk}$$

$$\longrightarrow \frac{1}{2\pi}\int_{\mathbb{R}^3} dz\, t_{abc}\,\underbrace{\partial^i_x}_{\text{acting only in b-direction}}$$



$$\downarrow$$

$$\int_{S^1} ds\, R^\alpha_{a\beta}\,\dot\gamma^i(s)$$

(−) sign for each red loop.

By a bit of a miracle, this boils down to a configuration space integral, which in itself, can be reduced to a pre-image count.

... But I run out of steam for tonight...

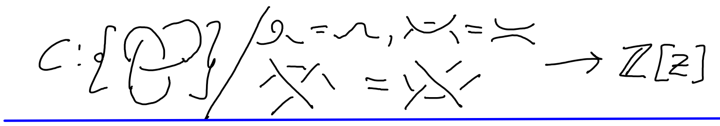
Banks like knots.

Lecture 3 Handout

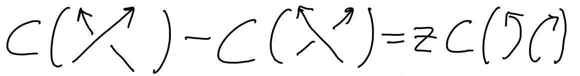
The Basics of Finite-Type Invariants of Knots

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

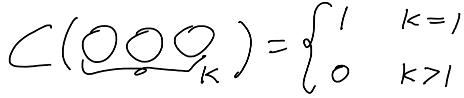
Definition. A knot invariant is any function whose domain is {knots}. Really, we mean a computable function whose target space is understandable; e.g.



Example. The Conway polynomial is given by



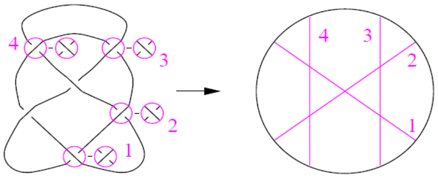
and

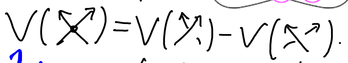


Exercise. Pick your favourite bank and compute the Conway polynomial of its logo.

INTERLINK

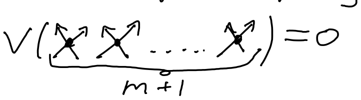
Definition. Any $V : \{\text{knots}\} \to$ Abelian Group A can be extended to "knots w/ double points" using





(Think "differentiation")

Definition. V is of type m if always

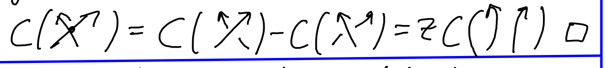


(think "polynomial")

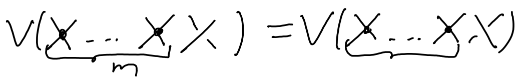
Conjecture. Finite type invariants separate knots.

Theorem. If $C(K)=\sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

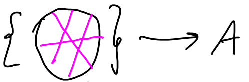
Proof.



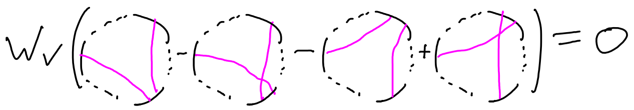
Let V be of type m; then $V^{(m)}$ is constant:



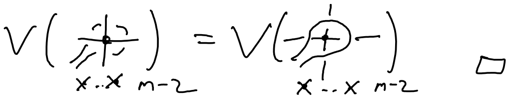
So $W_V := V^{(m)} = V|_{m\text{-singular knots}}$ is really a function on m-chord diagrams: W_V:



Claim. W_V satisfies the 4T relation:



Proof.



Exercise for Lecture 2. Use $\int_{\mathbb{R}} e^{-x^2/2} = \sqrt{2\pi}$, Fubini's theorem, and polar coordinates to compute $\int_{\mathbb{R}^n} e^{-\|x\|^2/2} d^n x$ in two different ways and hence to deduce the volume of S^{n-1}, the $(n-1)$-dimensional sphere.

Exercise. 1. Determine the "weight system" W_{V_m} of the m-th coefficient of the Conway polynomial and verify that is satisfies 4T.

2. Learn somewhere about the Jones polynomial, and do the same for its coefficients.

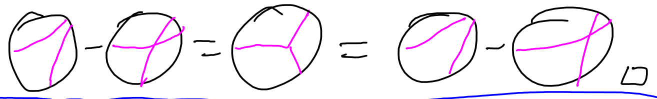
Theorem. (The Fundamental Theorem) Every "weight system", i.e. every linear functional W on $\mathcal{A} := \{\text{chord diagrams}\}/4T$ is the m'th derivative of a type m invariant: $\forall W \, \exists V$ s.t. $W = W_V$

M. Kontsevich

m	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}^r_m$	1	0	1	1	3	4	9	14	27	44	80	132	232
$\dim \mathcal{A}_m$	1	1	2	3	6	10	19	33	60	104	184	316	548
$\dim \mathcal{P}_m$	0	1	1	1	2	3	5	8	12	18	27	39	55

Theorem. $\mathcal{A}^{today} \cong \mathcal{A}^{Monday}$

Proof



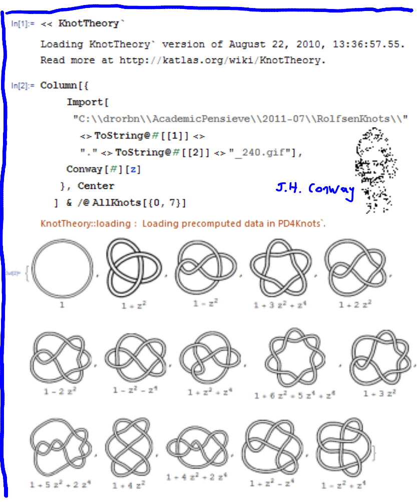
Proposition. The fundamental thm holds iff there exists an expansion $Z : \mathcal{K} \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then $Z(K) = D_K +$ higher degrees.

Proof.

$$\begin{tikzcd} \mathcal{K} \arrow[r, "Z"] \arrow[dr, "V"'] & \mathcal{A} \arrow[d, "W"] \\ & \mathbb{Q} \end{tikzcd}$$

□

Also see my old paper, "On the Vassiliev knot invariants" (google will find..)



```
In[1]:= << KnotTheory`

    Loading KnotTheory` version of August 22, 2010, 13:36:57.55.
    Read more at http://katlas.org/wiki/KnotTheory.

In[2]:= Column[{
          Import[
           "C:\\drorbn\\AcademicPensieve\\2011-07\\RolfsenKnots\\"
            <> ToString@#[[1]] <>
            "." <> ToString@#[[2]] <> "_240.gif"],
          Conway[#][z]
         }, Center
        ] & /@ AllKnots[{0, 7}]

    KnotTheory::loading : Loading precomputed data in PD4Knots`.
```

Lecture 4 Handout

Low and High Algebra in the "u" Case

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

The big picture, "u" Case.

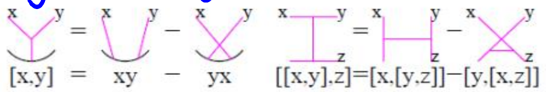
Topology — Combinatorics

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "Z", "\text{expansion}"'] & \mathcal{A} \arrow[r, bend left, "\text{choose } \mathfrak{g}, R"] \arrow[dr, bend right, "\text{choose } \mathfrak{g}"'] & \mathbb{Q} \\
 & & \mathcal{U}(\mathfrak{g})
\end{tikzcd}$$

$\left\{\begin{matrix}\text{knots}\\ \text{etc.}\end{matrix}\right\}$ $\quad \left\{\begin{matrix}\text{chord}\\ \text{diagrams}\end{matrix}\right\} \Big/ \begin{matrix}\text{IHX}\\ \text{STU}\\ \text{4T}\end{matrix}$

$\underbrace{\qquad\qquad\qquad}_{\text{high algebra}}\underbrace{\qquad\qquad\qquad}_{\text{low algebra}}$

Very low algebra.



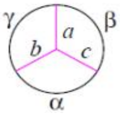
$[x,y] = xy - yx \qquad [[x,y],z]=[x,[y,z]]-[y,[x,z]]$


Sophus Lie

More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c \rangle \qquad X_a v_\beta = \sum_{\cancel{\beta}\gamma} r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak{g},R}:$ 

$$\longrightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$$

Excrsice. Find a fast method to find $W_{\mathfrak{g},R}(D)$ when $\mathfrak{g} = \mathfrak{gl}_n$, $R = \mathbb{R}^n$? Is it related to the Conway polynomial?

Universal Representation Theory.

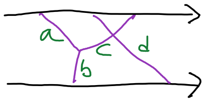
Inspired by $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$, set $\mathcal{U}(\mathfrak{g}) = \langle \text{words in } \mathfrak{g} \rangle / [x,y] = xy - yx$

- * Every rep of $\mathfrak{g}$ extends to $\mathcal{U}(\mathfrak{g})$.
- * $\exists\, \Delta : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ by "word splitting", as must be for $R_1 \otimes R_2$.

Exercise. With $\mathfrak{g} = \langle x, y \rangle / [x,y] = x$, determine $\mathcal{U}(\mathfrak{g})$. Guess a generalization.

Low algebra.

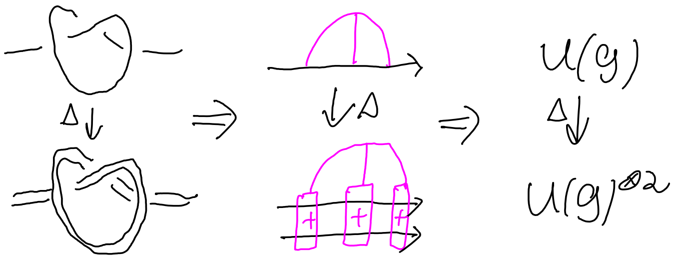
$\mathcal{A}(\uparrow\uparrow) \to \mathcal{U}(\mathfrak{g})^{\otimes 2}$ via



$$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} X_a X_d X_c \\ \otimes \\ X_b X_d \end{pmatrix}$$

& likewise, $\mathcal{A}(\uparrow_n) \to \mathcal{U}(\mathfrak{g})^{\otimes n} \Rightarrow$ $\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?

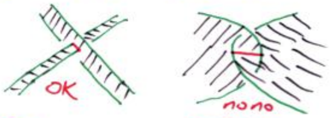


$$\begin{tikzcd}
\mathcal{U}(\mathfrak{g}) \arrow[d, "\Delta"'] \\
\mathcal{U}(\mathfrak{g})^{\otimes 2}
\end{tikzcd}$$

A "Homomorphic Expansion" $Z : \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is **High Algebra.**

$$\{\text{Ribbon knots}\} \overset{\uparrow \text{ more or less}}{=} \left\{ u\gamma : \begin{matrix} \gamma \in \mathcal{K}(\bigcirc\!\!-\!\!\bigcirc) \\ d\gamma = \bigcirc\bigcirc \end{matrix} \right\}$$

where Ribbon means 

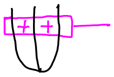
and

Algebraic knot theory:

$$\begin{tikzcd}
 & \mathcal{K}(\bigcirc\bigcirc) \arrow[r, "Z"] & \mathcal{A}(\bigcirc\bigcirc) \ni \bigcirc\bigcirc \\
\mathcal{K}(\bigcirc\!\!-\!\!\bigcirc) \arrow[ur, "d"] \arrow[r, "Z"] \arrow[dr, "u"'] & & \mathcal{A}(\bigcirc\!\!-\!\!\bigcirc) \arrow[u, "d"'] \arrow[d, "u"] \\
 & \mathcal{K}(\bigcirc\!\!=\!\!\bigcirc) \arrow[r, "Z"] & \mathcal{A}(\bigcirc)
\end{tikzcd}$$

So

$$Z\left(\left\{\begin{matrix}\text{Ribbon}\\ \text{knots}\end{matrix}\right\}\right) \subset \{ u\alpha : d\alpha = Z(\bigcirc\bigcirc) \} \subset \mathcal{A}(\bigcirc\!\!-\!\!\bigcirc)$$

VI  $= 0$, follows from  $=$ 

An Associator: Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{tikzcd}
((AB)C)D \arrow[rr, "(\Delta 11)\Phi"] \arrow[d, "\Phi 1"'] & & (AB)(CD) \arrow[d, "(11\Delta)\Phi"] \\
(A(BC))D \arrow[dr, "(1\Delta 1)\Phi"'] & & A(B(CD)) \\
 & A((BC)D) \arrow[ur, "1\Phi"'] &
\end{tikzcd}$$

$$\Phi 1 \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$$

V.G. Drinfel'd 

The hexagon? Never heard of it.

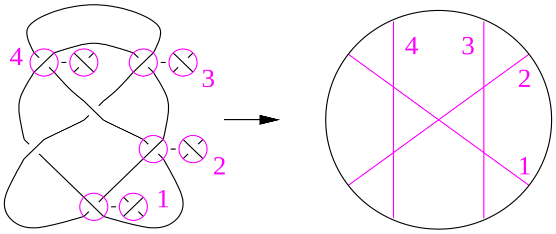
See Also. B-N & Dancso, arXiv: 1103.1896

Knotted Trivalent Graphs, Tetrahedra and Associators

HUJI Topology and Geometry Seminar, November 16, 2000

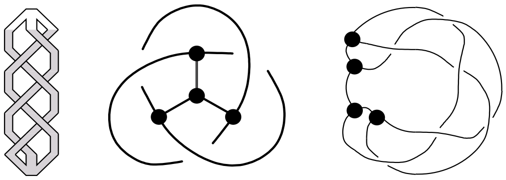
Dror Bar−Natan

Goal: Z:{knots}−>{chord diagrams}/4T so that

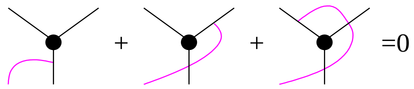


The Miller Institute knot

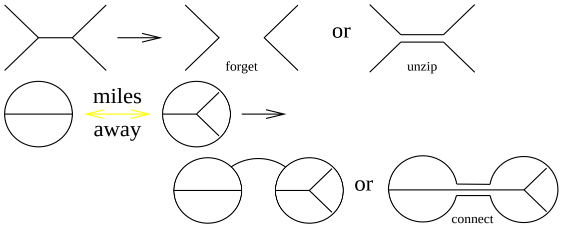
Extend to Knotted Trivalent Graphs (KTG's):

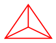


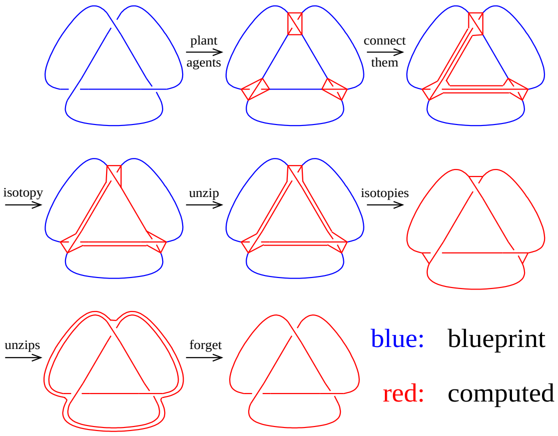
Need a new relation:



Easy, powerful moves:

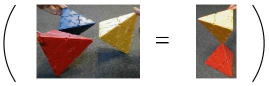


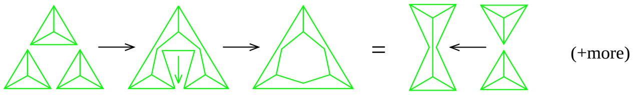
Using moves, KTG is generated by ribbon twists and the tetrahedron  :



blue: blueprint

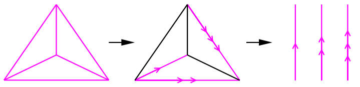
red: computed

Modulo the relation(s): 

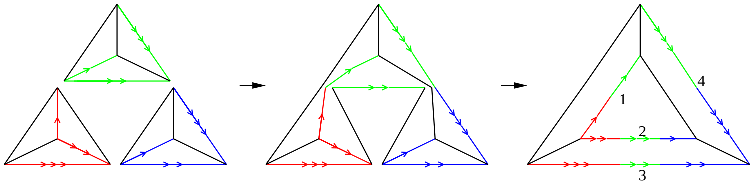


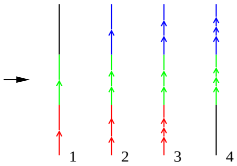
Claim. With $\Phi := Z(\triangle)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi Hopf algebras.

Proof.

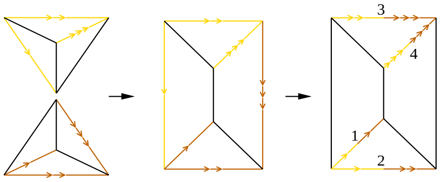


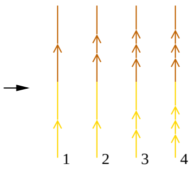
$:= \Phi \in \mathcal{A}(\uparrow_3)$





$= (\Phi \otimes 1) \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot (1 \otimes \Phi) \in \mathcal{A}(\uparrow_4)$





$= (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi) \in \mathcal{A}(\uparrow_4)$

Further directions:

1. Relations with perturbative Chern−Simons theory.
2. Relations with the theory of 6j symbols
3. Relations with the Turaev−Viro invariants.
4. Can this be used to prove the Witten asymptotics conjecture?
5. Does this extend/improve Drinfel'd's theory of associators?

This handout is at http://www.ma.huji.ac.il/~drorbn/Talks/HUJI−001116

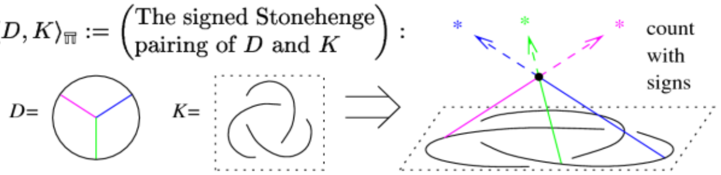
More at Talks/HUJI-001116/

Lecture 5 Extras

Review Material (mostly)

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

$\langle D, K\rangle_{\text{Ⓜ}} :=$ (The signed Stonehenge pairing of D and K) :



Thus we consider the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\text{Ⓜ}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

Theorem. Modulo Relations, *Z(K)* is a knot invariant!

When deforming, catastrophes occur when:

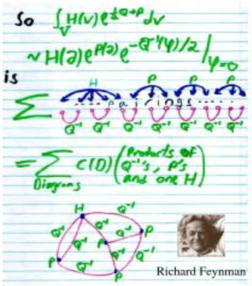
A plane moves over an intersection point – Solution: Impose IHX,

An intersection line cuts through the knot – Solution: Impose STU,

The Gauss curve slides over a star – Solution: Multiply by

(see below)

(similar argument)

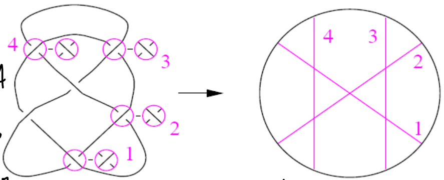


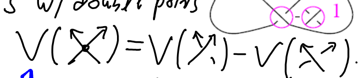
It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak g\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

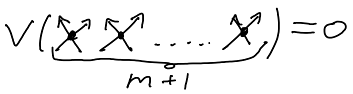
$$\to \sum_{D:\text{ Feynman diagram}} W_{\mathfrak g}(D) \sum \mathcal{E}(D) \to \sum_{D:\text{ Feynman diagram}} D \sum \mathcal{E}(D)$$

Definition. Any $V: \{\text{knots}\} \to$ Abelian Group A can be extended to "knots w/ double points"



using . (Think "differentiation")

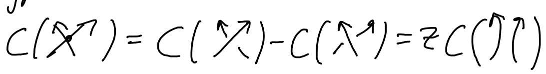
Definition. V is of type m if always



(think "polynomial")

Conjecture. Finite type invariants separate knots.

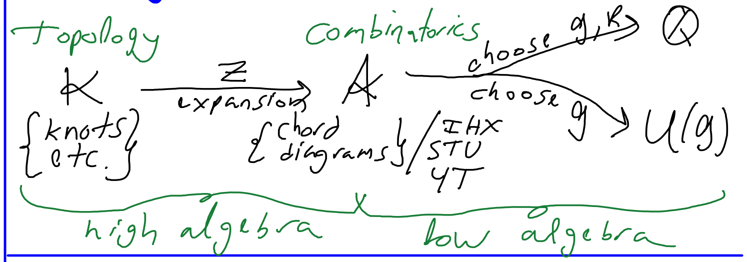
Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

Proof. 

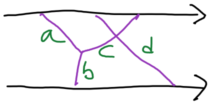
Proposition. The Fundamental thm holds iff there exists an expansion $Z: \mathcal{K} \longrightarrow \hat{\mathcal{A}}$ s.t. if K is m-singular, then

$$Z(K) = D_K + \text{higher degrees}$$

The big picture, "u" case.



Low algebra. $\mathcal{A}(\uparrow\uparrow) \to U(\mathfrak g)^{\otimes 2}$ via

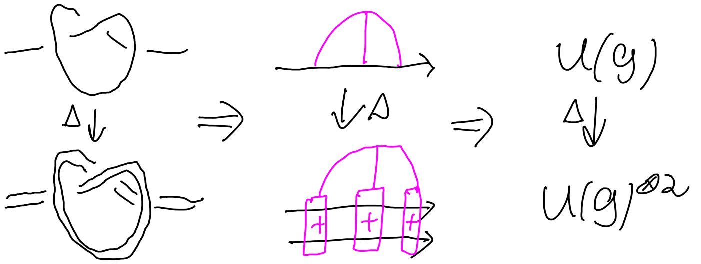


$$\longrightarrow \sum_{a\text{-}d} f_{abc} \begin{pmatrix} x_a x_d x_c \\ \otimes \\ x_b x_d \end{pmatrix}$$

& likewise, $\mathcal{A}(\uparrow_n) \longrightarrow U(\mathfrak g)^{\otimes n} \Rightarrow$

$\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?



A "Homomorphic Expansion" $Z: \mathcal{K} \to \mathcal{A}$

is an expansion that intertwines all relevant algebraic ops. If $\mathcal{K}$ is finitely presented, finding Z is High Algebra.

An Associator:

Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak g)^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{tikzcd}
((AB)C)D \arrow[r, "(\Delta 11)\Phi"] \arrow[d, "\Phi 1"'] & (AB)(CD) \arrow[d, "(11\Delta)\Phi"] \\
(A(BC))D \arrow[dr, "(1\Delta 1)\Phi"'] & A(B(CD)) \\
& A((BC)D) \arrow[u, "1\Phi"']
\end{tikzcd}$$

$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

V.G. Drinfel'd

The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv:1103.1896

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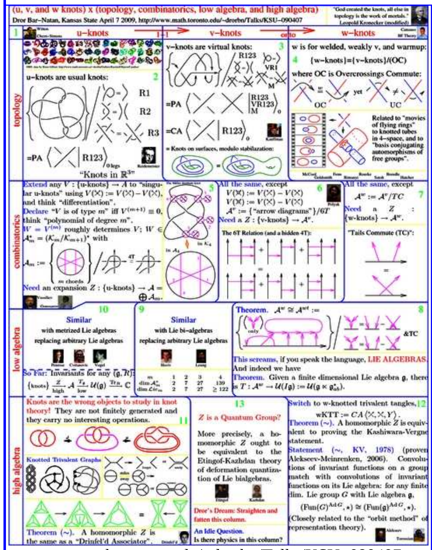
Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots

Dror Bar−Natan, Bonn August 2009, http://www.math.toronto.edu/~drorbn/Talks/Bonn−0908

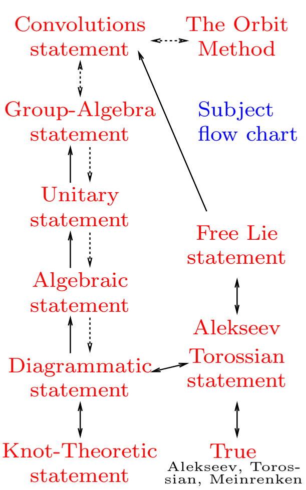
Disclaimer: Rough edges remain!

"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)



www.math.toronto.edu/~drorbn/Talks/KSU−090407

The Bigger Picture...



Kashiwara

Vergne

Alekseev

Torossian

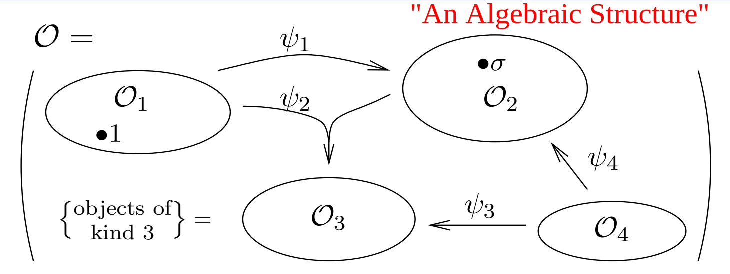
Homomorphic expansions for a filtered algebraic structure $\mathcal{K}$:

$$\begin{array}{ccccccccc} \text{ops}\circlearrowright\mathcal{K} = & \mathcal{K}_0 & \supset & \mathcal{K}_1 & \supset & \mathcal{K}_2 & \supset & \mathcal{K}_3 & \supset \dots \\ \Downarrow & & & & & \downarrow Z & & & \\ \text{ops}\circlearrowright \operatorname{gr}\mathcal{K} := & \mathcal{K}_0/\mathcal{K}_1 & \oplus & \mathcal{K}_1/\mathcal{K}_2 & \oplus & \mathcal{K}_2/\mathcal{K}_3 & \oplus & \mathcal{K}_3/\mathcal{K}_4 & \oplus \dots \end{array}$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \to \operatorname{gr}\mathcal{K}$ that "covers" the identity on $\operatorname{gr}\mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant "extra" operations.

Filtered algebraic structures are cheap and plenty. In any $\mathcal{K}$, allow formal linear combinations, let $\mathcal{K}_1$ be the ideal generated by differences (the "augmentation ideal"), and let $\mathcal{K}_m := \langle(\mathcal{K}_1)^m\rangle$ (using all available "products").

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Example: Pure Braids. PB_n is generated by x_{ij}, "strand i goes around strand j once", modulo "Reidemeister moves". $A_n := \operatorname{gr} PB_n$ is generated by $t_{ij} := x_{ij} - 1$, modulo the $4T$ relations $[t_{ij}, t_{ik} + t_{jk}] = 0$ (and some lesser ones too). Much happens in A_n, including the Drinfel'd theory of associators.

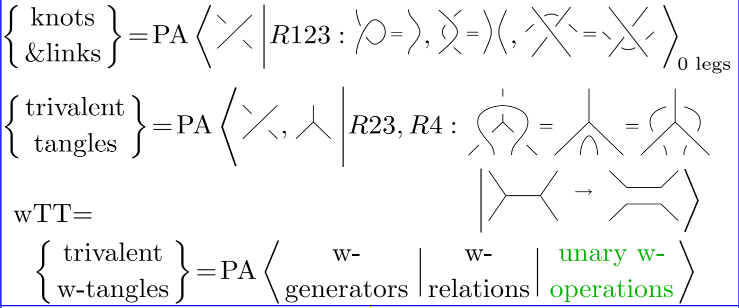
Our case(s).

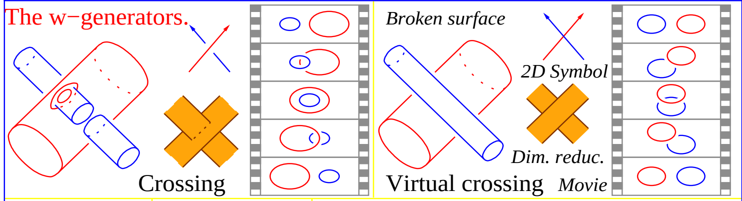
$$\mathcal{K} \xrightarrow[\substack{\text{solving finitely many}\\ \text{equations in finitely}\\ \text{many unknowns}}]{Z\text{: high algebra}} \mathcal{A} := \operatorname{gr}\mathcal{K} \xrightarrow[\substack{\text{low algebra: pic-}\\ \text{tures represent}\\ \text{formulas}}]{\substack{\text{given a "Lie"}\\ \text{algebra } \mathfrak{g}}} \text{"}\mathcal{U}(\mathfrak{g})\text{"}$$

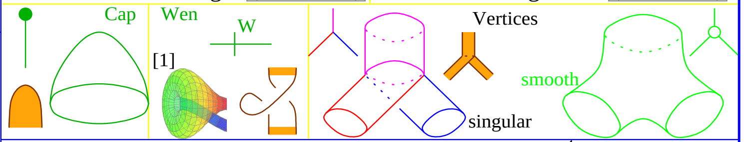
$\mathcal{K}$ is knot theory or **topology**; $\operatorname{gr}\mathcal{K}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.

What are w-Trivalent Tangles?

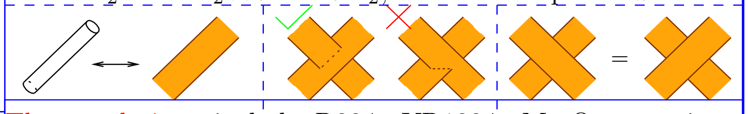
(PA :=Planar Algebra)



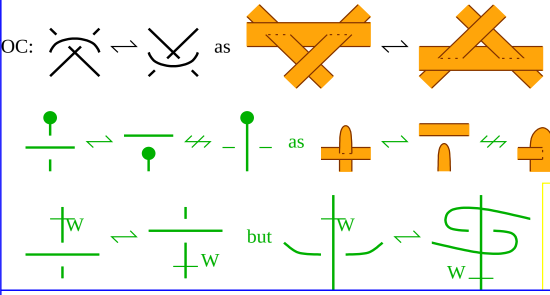




A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



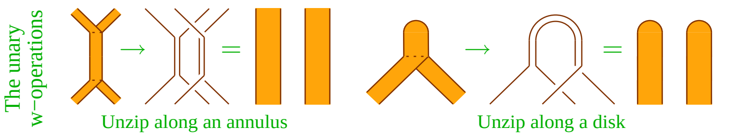
The w-relations include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



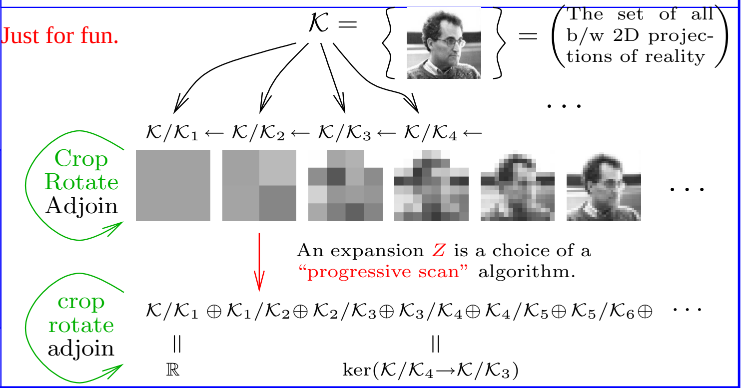
Challenge. Do the Reidemeister!

Reidemeister

Winter



Just for fun.

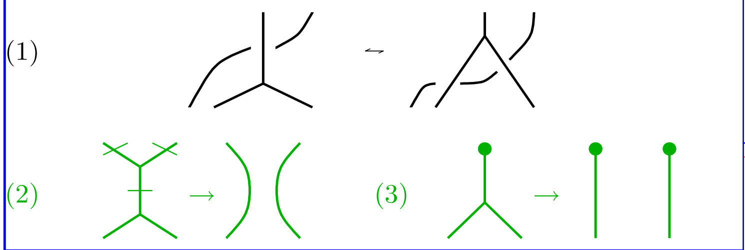


[1] http://qlink.queensu.ca/~4lb11/interesting.html 29/5/10, 8:42am
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

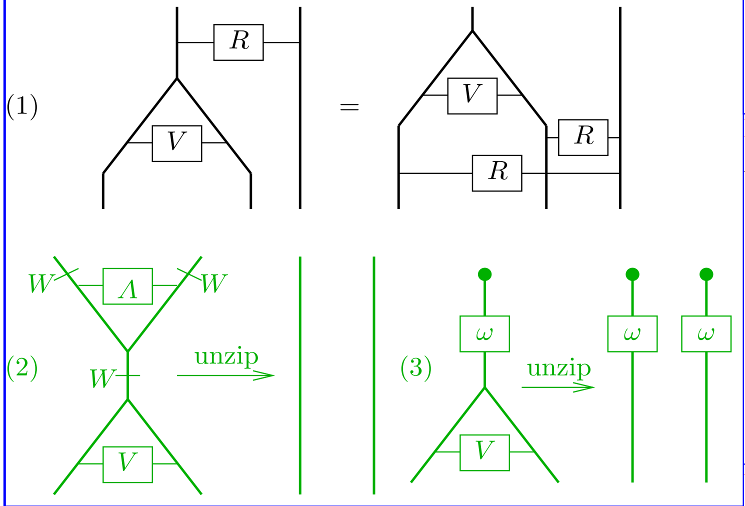
Video and more at Talks/Bonn-0908/

Convolutions on Lie Groups and Lie Algebras and Ribbon 2−Knots, Page 2

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \overrightarrow{\vdash\!\dashv} \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{\mathcal{U}}(I\mathfrak{g})$, with W the automorphism of $\hat{\mathcal{U}}(I\mathfrak{g})$ induced by flipping the sign of $\mathfrak{g}^*$, with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{\mathcal{U}}(I\mathfrak{g}) \otimes \hat{\mathcal{U}}(\mathfrak{g})$ there exist $\omega \in \hat{\mathcal{S}}(\mathfrak{g}^*)$ and $V \in \hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{\mathcal{U}}(I\mathfrak{g})^{\otimes 2} \otimes \hat{\mathcal{U}}(\mathfrak{g})$

(2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \mathrm{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\mathrm{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x}\widehat{e^y}V$ (allowing $\hat{\mathcal{U}}(\mathfrak{g})$-valued functions)

(2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x\omega_y$

Group-Algebra statement. There exists $\omega^2 \in \mathrm{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \mathrm{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $\omega^2 = j^{1/2}$)

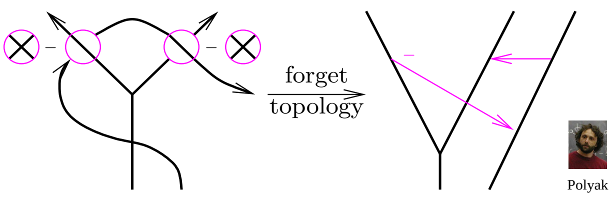
$$\iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_{x+y}e^{x+y} = \iint_{\mathfrak{g}\times\mathfrak{g}} \phi(x)\psi(y)\omega^2_x\omega^2_y e^x e^y.$$

(shhh, this is Duflo)

Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let $\mathfrak{g}$ be its Lie algebra, let $j : \mathfrak{g} \to \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \to G$, and let $\Phi : \mathrm{Fun}(G) \to \mathrm{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \mathrm{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

From wTT to $\mathcal{A}^w$. $\mathrm{gr}_m\,\mathrm{wTT} := \{m\text{−cubes}\}/\{(m+1)\text{−cubes}\}$:

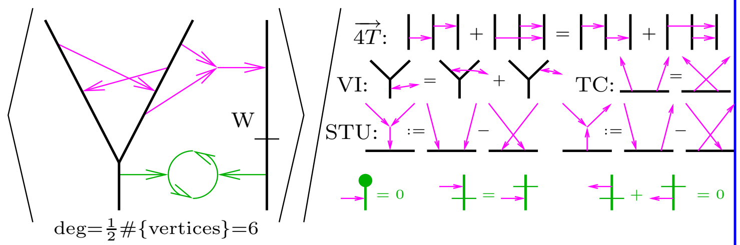



Vassiliev

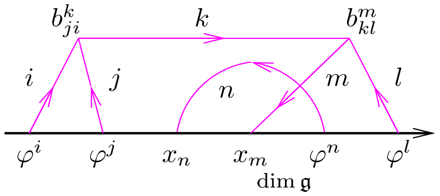

Polyak


Goussarov

w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ and with $[x_i, x_j] = \sum b^k_{ij}x_k$, we have $\mathcal{A}^w \to \mathcal{U}$ via




Penrose


Cvitanovic

$$\longrightarrow \sum_{i,j,k,l,m,n=1}^{\dim\mathfrak{g}} b^k_{ij}b^m_{kl}\varphi^i\varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary $\Longleftrightarrow$ Algebraic. The key is to interpret $\hat{\mathcal{U}}(I\mathfrak{g})$ as tangential differential operators on $\mathrm{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\mathrm{ad}\,x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \hat{\mathcal{U}}(I\mathfrak{g}) \to \hat{\mathcal{U}}(I\mathfrak{g})/\hat{\mathcal{U}}(\mathfrak{g}) = \hat{\mathcal{S}}(\mathfrak{g}^*)$ is "the constant term".

Unitary $\Longrightarrow$ Group-Algebra. $\iint \omega^2_{x+y}e^{x+y}\phi(x)\psi(y)$

$$= \langle \omega_{x+y}, \omega_{x+y}e^{x+y}\phi(x)\psi(y)\rangle = \langle V\omega_{x+y}, Ve^{x+y}\phi(x)\psi(y)\omega_{x+y}\rangle$$
$$= \langle \omega_x\omega_y, e^xe^yV\phi(x)\psi(y)\omega_{x+y}\rangle = \langle \omega_x\omega_y, e^xe^y\phi(x)\psi(y)\omega_x\omega_y\rangle$$
$$= \iint \omega^2_x\omega^2_y e^xe^y\phi(x)\psi(y).$$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \to A$ is multiplicative then $(\mathrm{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie $(G, \mathfrak{g})$,

$$\begin{tikzcd}
(\mathfrak{g},+) \ni x \arrow[r, "\tau_0=\exp_{\mathcal{S}}"] \arrow[d, "\exp_G"] \arrow[dr, "\exp_{\mathcal{U}}"] & e^x \in \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] & \text{so} & \mathrm{Fun}(\mathfrak{g}) \arrow[r, "L_0"] \arrow[d, "\Phi^{-1}"] & \hat{\mathcal{S}}(\mathfrak{g}) \arrow[d, "\chi"] \\
(G,\cdot) \ni e^x \arrow[r, "\tau_1"] & e^x \in \hat{\mathcal{U}}(\mathfrak{g}) & & \mathrm{Fun}(G) \arrow[r, "L_1"] & \hat{\mathcal{U}}(\mathfrak{g})
\end{tikzcd}$$

with $L_0\psi = \int\psi(x)e^x dx \in \hat{\mathcal{S}}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int\psi(x)e^x \in \hat{\mathcal{U}}(\mathfrak{g})$. Given $\psi_i \in \mathrm{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{\mathcal{U}}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$\star$ in G : $\iint \psi_1(x)\psi_2(y)e^xe^y$ $\star$ in $\mathfrak{g}$: $\iint \psi_1(x)\psi_2(y)e^{x+y}$

We skipped...

- The Alexander polynomial and Milnor numbers.
- u-Knots, Alekseev-Torossian, and Drinfel'd associators.
- v-Knots, quantum groups and Etingof-Kazhdan.
- BF theory and the successful religion of path integrals.
- The simplest problem hyperbolic geometry solves.

Video and more at `Talks/Bonn-0908/`