# Trees and Wheels and Balloons and Hoops

Dror Bar-Natan, Nha Trang, May 2013
ωεβ:=http://www.math.toronto.edu/~drorbn/Talks/NhaTrang-1305

## 15 Minutes on Algebra

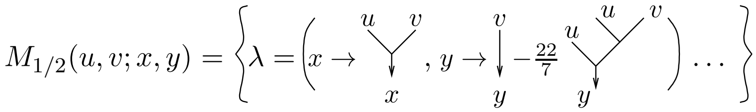
Let $T$ be a finite set of "tail labels" and $H$ a finite set of "head labels". Set

$$M_{1/2}(T;H) := FL(T)^H,$$

"$H$-labeled lists of elements of the degree-completed free Lie algebra generated by $T$".

$$FL(T) = \left\{2t_2 - \frac{1}{2}[t_1,[t_1,t_2]] + \dots\right\} \Big/ \left(\begin{matrix}\text{anti-symmetry}\\ \text{Jacobi}\end{matrix}\right)$$

... with the obvious bracket.




**Operations $M_{1/2} \to M_{1/2}$.** newspeak!

**Tail Multiply $tm^{uv}_w$** is $\lambda \mapsto \lambda /\!\!/ (u,v \to w)$, satisfies "meta-associativity", $tm^{uv}_u /\!\!/ tm^{uw}_u = tm^{vw}_v /\!\!/ tm^{uv}_u$.

**Head Multiply $hm^{xy}_z$** is $\lambda \mapsto (\lambda \setminus \{x,y\}) \cup (z \to \mathrm{bch}(\lambda_x, \lambda_y))$, where

$$\mathrm{bch}(\alpha,\beta) := \log(e^\alpha e^\beta) = \alpha + \beta + \frac{[\alpha,\beta]}{2} + \frac{[\alpha,[\alpha,\beta]] + [[\alpha,\beta],\beta]}{12} + \dots$$

satisfies $\mathrm{bch}(\mathrm{bch}(\alpha,\beta),\gamma) = \log(e^\alpha e^\beta e^\gamma) = \mathrm{bch}(\alpha, \mathrm{bch}(\beta,\gamma))$ and hence meta-associativity, $hm^{xy}_x /\!\!/ hm^{xz}_x = hm^{yz}_y /\!\!/ hm^{xy}_x$.

**Tail by Head Action $tha^{ux}$** is $\lambda \mapsto \lambda /\!\!/ RC^{\lambda_x}_u$, where $C^{-\gamma}_u\colon FL \to FL$ is the substitution $u \to e^{-\gamma} u e^{\gamma}$, or more precisely,

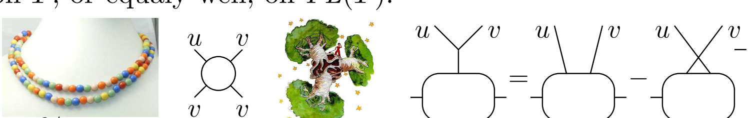
$$C^{-\gamma}_u\colon u \to e^{-\operatorname{ad}\gamma}(u) = u - [\gamma,u] + \frac{1}{2}[\gamma,[\gamma,u]] - \dots,$$

and $RC^\gamma_u$ is the inverse of that. Note that $C^{\mathrm{bch}(\alpha,\beta)}_u = C^{\alpha /\!\!/ RC^{-\beta}_u}_u /\!\!/ C^\beta_u$ and hence "meta $u^{xy} = (u^x)^y$",

$$hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z,$$

and $tm^{uv}_w /\!\!/ C^{\gamma /\!\!/ tm^{uv}_w}_w = C^{\gamma /\!\!/ RC^{-\gamma}_v}_u /\!\!/ C^\gamma_v /\!\!/ tm^{uv}_w$ and hence "meta $(uv)^x = u^x v^x$", $tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w$.

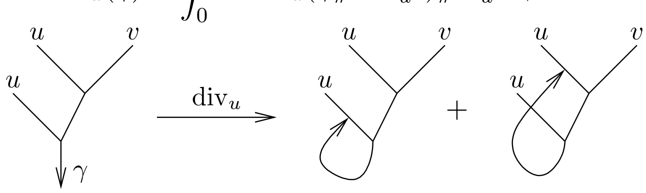
**Wheels.** Let $M(T;H) := M_{1/2}(T;H) \times CW(T)$, where $CW(T)$ is the (completed graded) vector space of cyclic words on $T$, or equaly well, on $FL(T)$:




**Operations.** On $M(T;H)$, define $tm^{uv}_w$ and $hm^{xy}_z$ as before, and $tha^{ux}$ by adding some $J$-spice:

$$(\lambda;\omega) \mapsto (\lambda, \omega + J_u(\lambda_x)) /\!\!/ RC^{\lambda_x}_u,$$

where $J_u(\gamma) := \int_0^1 ds\ \mathrm{div}_u(\gamma /\!\!/ RC^{s\gamma}_u) /\!\!/ C^{-s\gamma}_u$, and





Alekseev

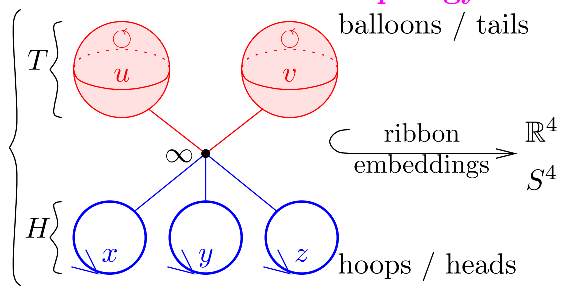

Torossian

**Theorem Blue.** All blue identities still hold.

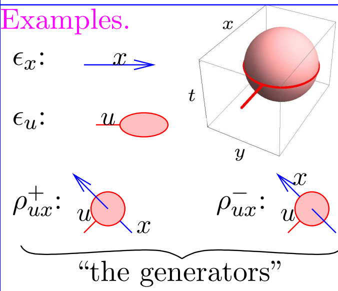
**Merge Operation.** $(\lambda_1;\omega_1) * (\lambda_2;\omega_2) := (\lambda_1 \cup \lambda_2; \omega_1 + \omega_2)$.

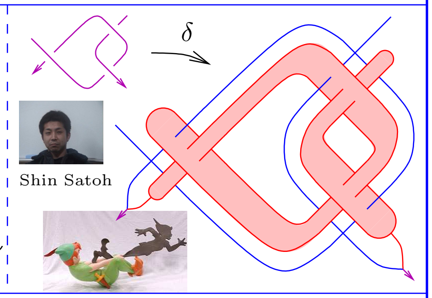
## 15 Minutes on Topology

$\mathcal{K}^{bh}(T;H)$. "Ribbon-knotted balloons and hoops"

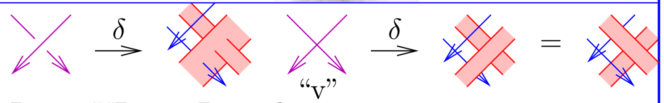



**Examples.**

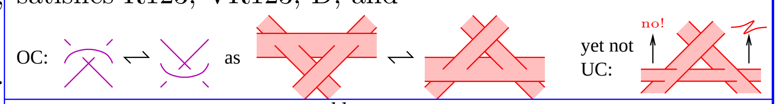



**More on δ**

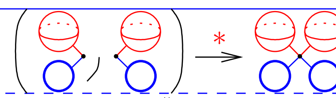



satisfies R123, VR123, D, and

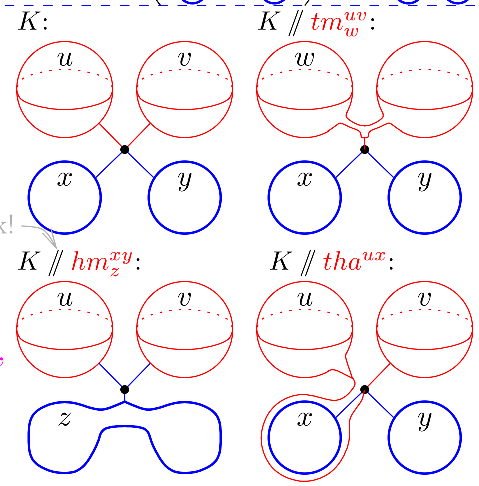



- δ injects u-knots into $\mathcal{K}^{bh}$ (likely u-tangles too).
- δ maps v-tangles to $\mathcal{K}^{bh}$; the kernel contains the above and conjecturally (Satoh), that's all.
- Allowing punctures and cuts, δ is onto.

**Operations** Punctures & Cuts | Connected Sums.



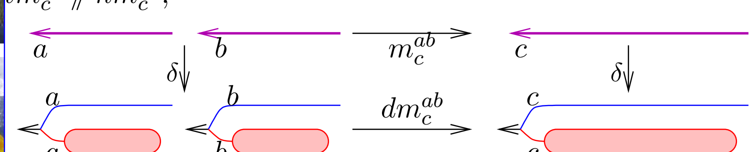

If $X$ is a space, $\pi_1(X)$ is a group, $\pi_2(X)$ is an Abelian group, and $\pi_1$ acts on $\pi_2$.

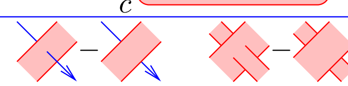

**Properties.**

- Associativities: $m^{ab}_a /\!\!/ m^{ac}_a = m^{bc}_b /\!\!/ m^{ab}_a$, for $m = tm, hm$.
- "$(uv)^x = u^x v^x$": $tm^{uv}_w /\!\!/ tha^{wx} = tha^{ux} /\!\!/ tha^{vx} /\!\!/ tm^{uv}_w$,
- "$u^{(xy)} = (u^x)^y$": $hm^{xy}_z /\!\!/ tha^{uz} = tha^{ux} /\!\!/ tha^{uy} /\!\!/ hm^{xy}_z$.

**Tangle concatenations → $\pi_1 \ltimes \pi_2$.** With $dm^{ab}_c := tha^{ab} /\!\!/ tm^{ab}_c /\!\!/ hm^{ab}_c$,



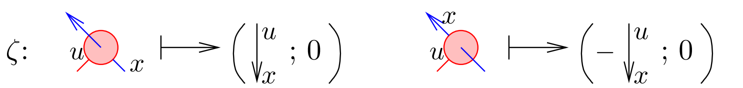

**Finite type** invariants make sense in the usual way, and "algebra" is (the primitive part of) "gr" of "topology".



# Trees and Wheels and Balloons and Hoops: Why I Care

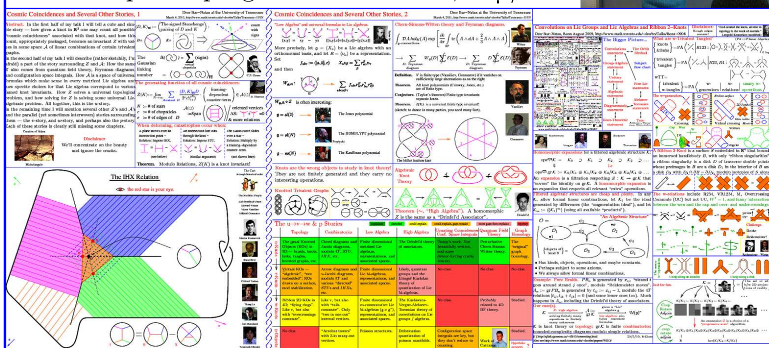
**Moral.** To construct an $M$-valued invariant $\zeta$ of (v-)tangles, and nearly an invariant on $\mathcal{K}^{bh}$, it is enough to declare $\zeta$ on the generators, and verify the relations that $\delta$ satisfies.

**The Invariant $\zeta$.** Set $\zeta(\epsilon_x)=(x\to 0;0)$, $\zeta(\epsilon_u)=(();0)$, and



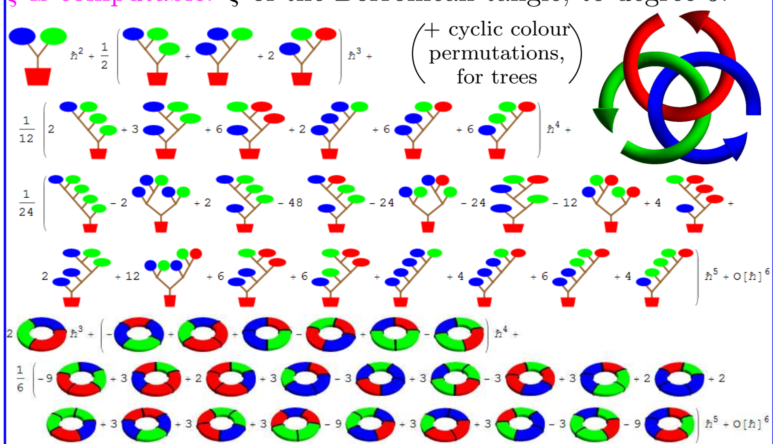

**Theorem.** $\zeta$ is (log of) the unique homomorphic universal finite type invariant on $\mathcal{K}^{bh}$.
(… and is the tip of an iceberg)

Paper in progress with Dancso, ωεβ/wko



See also ωεβ/tenn, ωεβ/bonn, ωεβ/swiss, ωεβ/portfolio

**$\zeta$ is computable!** $\zeta$ of the Borromean tangle, to degree 5:




**Tensorial Interpretation.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra (any!). Then there's $\tau: FL(T)\to \mathrm{Fun}(\oplus_T\mathfrak{g}\to\mathfrak{g})$ and $\tau: CW(T)\to\mathrm{Fun}(\oplus_T\mathfrak{g})$. Together, $\tau: M(T;H)\to \mathrm{Fun}(\oplus_T\mathfrak{g}\to\oplus_H\mathfrak{g})$, and hence

$$e^\tau: M(T;H)\to \mathrm{Fun}(\oplus_T\mathfrak{g}\to\mathcal{U}^{\otimes H}(\mathfrak{g})).$$

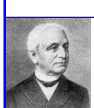
**$\zeta$ and BF Theory.** (See Cattaneo-Rossi, arXiv:math-ph/0210037) Let $A$ denote a $\mathfrak{g}$-connection on $S^4$ with curvature $F_A$, and $B$ a $\mathfrak{g}^*$-valued 2-form on $S^4$. For a hoop $\gamma_x$, let $\mathrm{hol}_{\gamma_x}(A)\in\mathcal{U}(\mathfrak{g})$ be the holonomy of $A$ along $\gamma_x$. For a ball $\gamma_u$, let $\mathcal{O}_{\gamma_u}(B)\in\mathfrak{g}^*$ be (roughly) the integral of $B$ (transported via $A$ to $\infty$) on $\gamma_u$.

Cattaneo

**Loose Conjecture.** For $\gamma\in\mathcal{K}(T;H)$,

$$\int \mathcal{D}A\mathcal{D}Be^{\int B\wedge F_A}\prod_u e^{\mathcal{O}_{\gamma_u}(B)}\bigotimes_x \mathrm{hol}_{\gamma_x}(A)=e^\tau(\zeta(\gamma)).$$

That is, $\zeta$ is a complete evaluation of the BF TQFT.

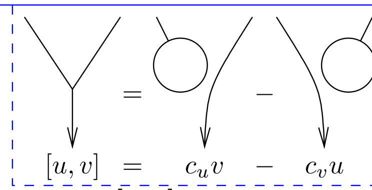


"God created the knots, all else in topology is the work of mortals."

Leopold Kronecker (modified)

www.katlas.org

**The $\beta$ quotient** is $M$ divided by all relations that universally hold when when $\mathfrak{g}$ is the 2D non-Abelian Lie algebra. Let $R=\mathbb{Q}[\![\{c_u\}_{u\in T}]\!]$ and $L_\beta:=R\otimes T$ with central $R$ and with $[u,v]=c_uv-c_vu$ for $u,v\in T$. Then $FL\to L_\beta$ and $CW\to R$. Under this,




$$\mu\to((\lambda_x);\omega)\quad\text{with } \lambda_x=\sum_{u\in T}\lambda_{ux}ux,\quad \lambda_{ux},\omega\in R,$$

$$\mathrm{bch}(u,v)\to\frac{c_u+c_v}{e^{c_u+c_v}-1}\left(\frac{e^{c_u}-1}{c_u}u+e^{c_u}\frac{e^{c_v}-1}{c_v}v\right),$$

if $\gamma=\sum\gamma_v v$ then with $c_\gamma:=\sum\gamma_vc_v$,

$$u\,/\!\!/\,RC_u^\gamma=\left(1+c_u\gamma_u\frac{e^{c_\gamma}-1}{c_\gamma}\right)^{-1}\left(e^{c_\gamma}u-c_u\frac{e^{c_\gamma}-1}{c_\gamma}\sum_{v\neq u}\gamma_vv\right),$$

$\mathrm{div}_u\gamma=c_u\gamma_u$, and $J_u(\gamma)=\log\left(1+\frac{e^{c_\gamma}-1}{c_\gamma}c_u\gamma_u\right)$, so $\zeta$ is formula-computable to all orders! Can we simplify?

**Repackaging.** Given $((x\to\lambda_{ux});\omega)$, set $c_x:=\sum_v c_v\lambda_{vx}$, replace $\lambda_{ux}\to\alpha_{ux}:=c_u\lambda_{ux}\frac{e^{c_x}-1}{c_x}$ and $\omega\to e^\omega$, use $t_u=e^{c_u}$, and write $\alpha_{ux}$ as a matrix. Get "$\beta$ calculus".

**$\beta$ Calculus.** Let $\beta(T;H)$ be

$$\left\{\begin{array}{c|ccc}\omega & x & y & \cdots\\\hline u & \alpha_{ux} & \alpha_{uy} & \cdot\\ v & \alpha_{vx} & \alpha_{vy} & \cdot\\ \vdots & \cdot & \cdot & \cdot\end{array}\;\middle|\;\begin{array}{l}\omega\text{ and the }\alpha_{ux}\text{'s are}\\ \text{rational functions in}\\ \text{variables }t_u,\text{ one for}\\ \text{each }u\in T.\end{array}\right\},$$

With Selmani, ωεβ/meta

$$tm^{uv}_w:\begin{array}{c|c}\omega&\cdots\\\hline u&\alpha\\ v&\beta\\ \vdots&\gamma\end{array}\mapsto\begin{array}{c|c}\omega&\cdots\\\hline w&\alpha+\beta\\ \vdots&\gamma\end{array},\qquad \begin{array}{c|c}\omega_1&H_1\\\hline T_1&\alpha_1\end{array}*\begin{array}{c|c}\omega_2&H_2\\\hline T_2&\alpha_2\end{array}=\begin{array}{c|cc}\omega_1\omega_2&H_1&H_2\\\hline T_1&\alpha_1&0\\ T_2&0&\alpha_2\end{array},$$

$$hm^{xy}_z:\begin{array}{c|ccc}\omega&x&y&\cdots\\\hline \vdots&\alpha&\beta&\gamma\end{array}\mapsto\begin{array}{c|cc}\omega&z&\cdots\\\hline \vdots&\alpha+\beta+\langle\alpha\rangle\beta&\gamma\end{array},$$

$$tha^{ux}:\begin{array}{c|cc}\omega&x&\cdots\\\hline u&\alpha&\beta\\ \vdots&\gamma&\delta\end{array}\mapsto\begin{array}{c|cc}\omega\epsilon&x&\cdots\\\hline u&\alpha(1+\langle\gamma\rangle/\epsilon)&\beta(1+\langle\gamma\rangle/\epsilon)\\ \vdots&\gamma/\epsilon&\delta-\gamma\beta/\epsilon\end{array},$$

where $\epsilon:=1+\alpha$, $\langle\alpha\rangle:=\sum_v\alpha_v$, and $\langle\gamma\rangle:=\sum_{v\neq u}\gamma_v$, and let

$$R^+_{ux}:=\begin{array}{c|c}1&x\\\hline u&t_u-1\end{array}\qquad R^-_{ux}:=\begin{array}{c|c}1&x\\\hline u&t_u^{-1}-1\end{array}.$$

On long knots, $\omega$ is the Alexander polynomial!

**Why happy?** An ultimate Alexander invariant: Manifestly polynomial (time and size) extension of the (multivariable) Alexander polynomial to tangles. Every step of the computation is the computation of the invariant of some topological thing (no fishy Gaussian elimination). *If there should be an Alexander invariant with a computable algebraic categorification, it is this one!*

See also ωεβ/regina, ωεβ/caen, ωεβ/newton.

May class: ωεβ/aarhus    Class next year: ωεβ/1350

Paper in progress: ωεβ/kbh

# The Most Important Missing Infrastructure Project in Knot Theory

January-23-12
10:12 AM

An "infrastructure project" is hard (and sometimes non-glorious) work that's done now and pays off later.

An example, and the most important one within knot theory, is the tabulation of knots up to 10 crossings. I think it precedes Rolfsen, yet the result is often called "the Rolfsen Table of Knots", as it is famously printed as an appendix to the famous book by Rolfsen. There is no doubt the production of the Rolfsen table was hard and non-glorious. Yet its impact was and is tremendous. Every new thought in knot theory is tested against the Rolfsen table, and it is hard to find a paper in knot theory that doesn't refer to the Rolfsen table in one way or another.

A second example is the Hoste-Thistlethwaite tabulation of knots with up to 17 crossings. Perhaps more fun to do as the real hard work was delegated to a machine, yet hard it certainly was: a proof is in the fact that nobody so far had tried to replicate their work, not even to a smaller crossing number. Yet again, it is hard to overestimate the value of that project: in many ways the Rolfsen table is "not yet generic", and many phenomena that appear to be rare when looking at the Rolfsen table become the rule when the view is expanded. Likewise, other phenomena only appear for the first time when looking at higher crossing numbers.

But as I like to say, knots are the wrong object to study in knot theory. Let me quote (with some variation) my own (with Dancso) "WKO" paper:

> Studying knots on their own is the parallel of studying cakes and pastries as they come out of the bakery - we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots (see [AKT-CFA]).
>
> The right objects for study in knot theory are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids (which are already well-studied and tabulated) and even more so tangles and tangled graphs.

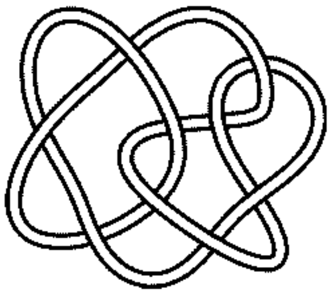
**Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical.** This will enable a great amount of testing and experimentation for which the grounds are now still missing. The existence of such a tabulation will greatly impact the direction of knot theory, as many tangle theories and issues that are now ignored for the lack of scope, will suddenly become alive and relevant. The overall influence of such a tabulation, if done right, will be comparable to the influence of the Rolfsen table.

Aside. What are tangles? Are they embedded in a disk? A ball? Do they have an "up side" and a "down side"? Are the strands oriented? Do we mod out by some symmetries or figure out the action of some symmetries? Shouldn't we also calculate the affect of various tangle operations (strand doubling and deletion, juxtapositions, etc.)? Shouldn't we also enumerate virtual tangles? w-tangles? Tangled graphs?

In my mind it would be better to leave these questions to the tabulator. Anything is better than nothing, yet good tabulators would try to tabulate the more general things from which the more special ones can be sieved relatively easily, and would see that their programs already contain all that would be easy to implement within their frameworks. Counting legs is easy and can be left to the end user. Determining symmetries is better done along with the enumeration itself, and so it should.

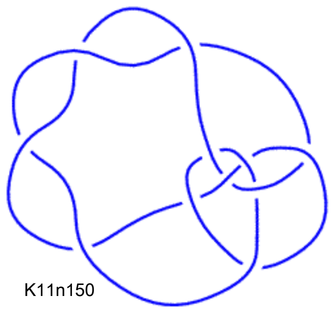
**An even better tabulation** should come with a modern front-end - a set of programs for basic manipulations of tangles, and a web-based "tangle atlas" for an even easier access.

Overall this would be a major project, well worthy of your time.



(KnotPlot image)

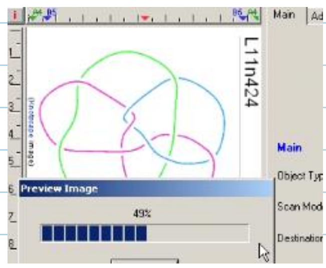
9_42 is Alexander Stoimenow's favourite




(Knotscape image)

The interchange of I-95 and I-695, northeast of Baltimore. (more)

From [AKT-CFA]




From [FastKh]


http://katlas.org/

(Source: http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2012-01/)

Pensieve header: Demo of the free-Lie meta-group-action structure for http://www.math.toronto.edu/~drorbn/Talks/NhaTrang-1305/.

```
Get["http://drorbn.net/AcademicPensieve/2013-05/FreeLie.m"];
Get["http://drorbn.net/AcademicPensieve/2013-05/muCalculus.m"];
```

```
u = ⟨"u"⟩; v = ⟨"v"⟩; BCH[u, v]@{6}
```

$\mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{\overline{uv}}{2},\ \frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{\overline{uv}v},\ \frac{1}{24}\overline{u\overline{\overline{uv}v}},$
$-\frac{1}{720}\overline{u\overline{u\overline{u\overline{uv}}}}+\frac{1}{180}\overline{u\overline{u\overline{\overline{uv}v}}}+\frac{1}{180}\overline{u\overline{\overline{\overline{uv}v}v}}+\frac{1}{120}\overline{\overline{uv}\overline{\overline{uv}v}}+\frac{1}{360}\overline{\overline{u\overline{uv}}\overline{uv}}-\frac{1}{720}\overline{\overline{\overline{\overline{uv}v}v}v},$
$-\frac{\overline{u\overline{u\overline{u\overline{\overline{uv}v}}}}}{1440}+\frac{1}{360}\overline{u\overline{u\overline{\overline{\overline{uv}v}v}}}+\frac{1}{240}\overline{u\overline{\overline{uv}\overline{\overline{uv}v}}}+\frac{1}{720}\overline{u\overline{\overline{u\overline{uv}}\overline{uv}}}-\frac{\overline{u\overline{\overline{\overline{\overline{uv}v}v}v}}}{1440}\Big]$

```
w = ⟨"w"⟩; Print /@ {BCH[BCH[u, v], w], BCH[u, BCH[v, w]]};
```

$\mathrm{LS}\Big[\overline{u}+\overline{v}+\overline{w},\ \frac{\overline{uv}}{2}+\frac{\overline{uw}}{2}+\frac{\overline{vw}}{2},$
$\frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{u\overline{uw}}+\frac{1}{3}\overline{u\overline{vw}}+\frac{1}{12}\overline{v\overline{vw}}+\frac{1}{12}\overline{\overline{uv}v}+\frac{1}{6}\overline{\overline{uw}v}+\frac{1}{12}\overline{\overline{uw}w}+\frac{1}{12}\overline{\overline{vw}w}\Big]$

$\mathrm{LS}\Big[\overline{u}+\overline{v}+\overline{w},\ \frac{\overline{uv}}{2}+\frac{\overline{uw}}{2}+\frac{\overline{vw}}{2},$
$\frac{1}{12}\overline{u\overline{uv}}+\frac{1}{12}\overline{u\overline{uw}}+\frac{1}{3}\overline{u\overline{vw}}+\frac{1}{12}\overline{v\overline{vw}}+\frac{1}{12}\overline{\overline{uv}v}+\frac{1}{6}\overline{\overline{uw}v}+\frac{1}{12}\overline{\overline{uw}w}+\frac{1}{12}\overline{\overline{vw}w}\Big]$

```
J_v[BCH[u, v]]@{4}
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$\mathrm{CWS}\Big[\overline{v},\ \overline{uv},\ \frac{\overline{uuv}}{2}-\frac{\overline{uvv}}{2},\ \frac{\overline{uuuv}}{6}-\frac{\overline{uuvv}}{4}-\frac{\overline{uvuv}}{2}+\frac{\overline{uvvv}}{6}\Big]$

Testing hm[x,y,z] // tha[u,z] ≡ tha[u,x] // tha[u,y] // hm[x,y,z]

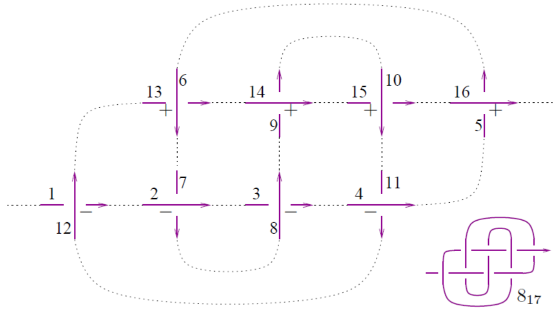
```
Print /@ {
  1 → (t1 = M[{x → MakeLieSeries[u + b[u, v]], y → MakeLieSeries[v + 2/3 b[u, v]]},
      MakeCWSeries[CW["uu"] + CW["uvv"]]]),
  2 → (t2 = t1 // hm[x, y, z] // tha[u, z]),
  3 → (t3 = t1 // tha[u, x] // tha[u, y] // hm[x, y, z]),
  4 → (t2 ≡ t3)};
```

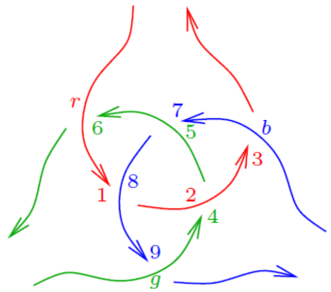
$1\to\mathrm{M}\Big[\Big\{x\to\mathrm{LS}[\overline{u},\overline{uv},0],\ y\to\mathrm{LS}\Big[\overline{v},\frac{2\,\overline{uv}}{3},0\Big]\Big\},\ \mathrm{CWS}[0,\overline{uu},\overline{uvv}]\Big]$

$2\to\mathrm{M}\Big[\Big\{z\to\mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{7\,\overline{uv}}{6},\ -\frac{5}{4}\overline{u\overline{uv}}-\frac{13}{12}\overline{\overline{uv}v}\Big]\Big\},\ \mathrm{CWS}\Big[\overline{u},\ \overline{uu}-\frac{5\,\overline{uv}}{3},\ \frac{\overline{uuv}}{2}+\frac{2\,\overline{uvv}}{3}\Big]\Big]$

$3\to\mathrm{M}\Big[\Big\{z\to\mathrm{LS}\Big[\overline{u}+\overline{v},\ \frac{7\,\overline{uv}}{6},\ -\frac{5}{4}\overline{u\overline{uv}}-\frac{13}{12}\overline{\overline{uv}v}\Big]\Big\},\ \mathrm{CWS}\Big[\overline{u},\ \overline{uu}-\frac{5\,\overline{uv}}{3},\ \frac{\overline{uuv}}{2}+\frac{2\,\overline{uvv}}{3}\Big]\Big]$

4 → True




## Demo 1 - The Knot $8_{17}$

```
μ1 = R⁻[12, 1] R⁻[2, 7] R⁻[8, 3] R⁻[4, 11] R⁺[16, 5] R⁺[6, 13] R⁺[14, 9] R⁺[10, 15]
```

$\mathrm{M}\Big[\Big\{1\to\mathrm{LS}[-\overline{c},0,0],\ 2\to\mathrm{LS}[0,0,0],\ 3\to\mathrm{LS}[-\overline{8},0,0],\ 4\to\mathrm{LS}[0,0,0],$
$5\to\mathrm{LS}[\overline{g},0,0],\ 6\to\mathrm{LS}[0,0,0],\ 7\to\mathrm{LS}[-\overline{2},0,0],\ 8\to\mathrm{LS}[0,0,0],\ 9\to\mathrm{LS}[\overline{e},0,0],$
$10\to\mathrm{LS}[0,0,0],\ 11\to\mathrm{LS}[-\overline{4},0,0],\ 12\to\mathrm{LS}[0,0,0],\ 13\to\mathrm{LS}[\overline{6},0,0],$
$14\to\mathrm{LS}[0,0,0],\ 15\to\mathrm{LS}[\overline{a},0,0],\ 16\to\mathrm{LS}[0,0,0]\Big\},\ \mathrm{CWS}[0,0,0]\Big]$

```
Do[μ1 = μ1 // dm[1, k, 1], {k, 2, 16}]; μ1[W]@{6}
```

$\mathrm{CWS}\Big[0,\ -\overline{11},\ 0,\ -\frac{31\,\overline{1111}}{12},\ 0,\ -\frac{1351\,\overline{111111}}{360}\Big]$

Compare with the Alexander polynomial:

```
Series[Log[-1/X^3 + 4/X^2 - 8/X + 11 - 8 X + 4 X^2 - X^3 /. X → e^x], {x, 0, 6}]
```

$-x^2-\frac{31\,x^4}{12}-\frac{1351\,x^6}{360}+O[x]^7$

## Demo 2 - The Borromean Tangle

```
μ2 = R⁻[r, 6] R⁺[2, 4] R⁻[g, 9] R⁺[5, 7] R⁻[b, 3] R⁺[8, 1];
(Do[μ2 = μ2 // dm[r, k, r], {k, 1, 3}]; Do[μ2 = μ2 // dm[g, k, g], {k, 4, 6}];
 Do[μ2 = μ2 // dm[b, k, b], {k, 7, 9}]; {μ2[r]@{4}, μ2[W]@{4}})
```

$\Big\{\mathrm{LS}\Big[0,\ \overline{bg},\ \frac{1}{2}\overline{b\overline{bg}}+\overline{\overline{bg}r}+\frac{1}{2}\overline{\overline{bg}g},$
$\frac{1}{6}\overline{b\overline{b\overline{bg}}}+\frac{1}{2}\overline{b\overline{\overline{bg}r}}+\frac{1}{2}\overline{b\overline{\overline{bg}g}r}+\frac{1}{4}\overline{b\overline{\overline{bg}g}}+\frac{1}{2}\overline{\overline{\overline{bg}r}r}+\frac{1}{6}\overline{\overline{\overline{bg}g}g}\Big],$
$\mathrm{CWS}\Big[0,\ 0,\ 2\,\overline{bgr},\ \overline{bbgr}-\overline{bgbr}+\overline{bggr}-\overline{bgrg}+\overline{bgrr}-\overline{brgr}\Big]\Big\}$