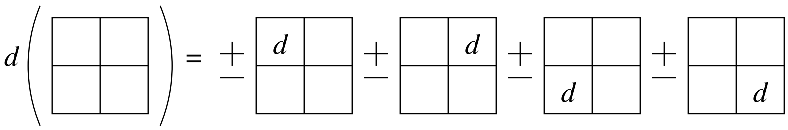
M. Khovanov 

The Khovanov–Rozansky Complex

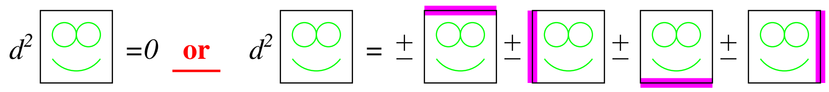
L. Rozansky 

Dror Bar–Natan at York University, November 2004, http://www.math.toronto.edu/~drorbn/Talks/York–041108/

Local differentials:

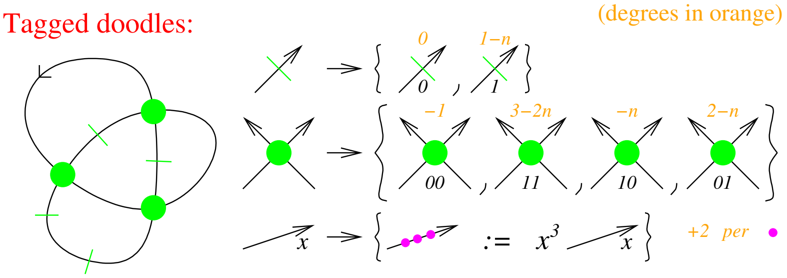


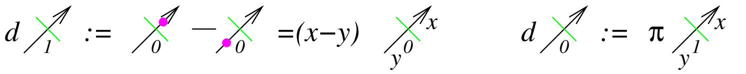
where



Tagged doodles:

(degrees in orange)






```
In[1]:= n = 2; π[i_, j_] := Cancel[(x_i^(n+1) - x_j^(n+1))/(x_i - x_j)]; π[1,2]
Out[1]= x1^2 + x1 x2 + x2^2
In[2]:= L = ( 0        x1 - x2 )
            ( π[1,2]   0       );
        Expand[L.L] // MatrixForm
Out[3]//MatrixForm=
( x1^3 - x2^3   0           )
( 0             x1^3 - x2^3 )
```

Set $L=d$ (deg $d = n+1$)

Matrix factorizations:

$$D = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$$

$$AB = BA = \omega I$$

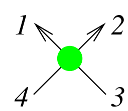
$$\begin{tikzcd} M^0 \arrow[r, "A"] \arrow[d, shift right, "U^0"'] \arrow[d, shift left, "V^0"] & M^1 \arrow[r, "B"] \arrow[dl, "h^1"'] \arrow[d, shift right, "U^1"'] \arrow[d, shift left, "V^1"] & M^0 \arrow[dl, "h^0"'] \arrow[d, shift right, "U^0"'] \arrow[d, shift left, "V^0"] \\ N^0 \arrow[r, "A'"] & N^1 \arrow[r, "B'"] & N^0 \end{tikzcd}$$

A category, with "complexes", morphisms, homotopies, direct sums and tensor products.

"God created the knots, all else in topology is the work of man."

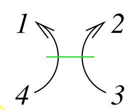
Leopold Kronecker (modified)

Likewise, set $Q=d$ with:



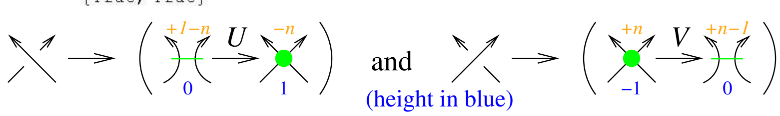

```
In[4]:= Q := ( 0    0    v1   v2  )
             ( 0    0    u2   -u1 )
             ( u1   v2   0    0   )
             ( u2   -v1  0    0   );
        {v1, v2} = {x1 + x2 - x3 - x4, x1 x2 - x3 x4};
In[6]:= g[s_, p_] :=
          s^(n+1) + (n+1) Sum[(-1)^i/i Binomial[n - i, i - 1] s^(n+1-2i) p^i, {i, 1, (n+1)/2}];
        g[x + y, x y] // Expand
Out[6]= x^3 + y^3
In[7]:= {u1, u2} =
          Cancel[{(g[x1 + x2, x1 x2] - g[x3 + x4, x1 x2])/v1,
                  (g[x3 + x4, x1 x2] - g[x3 + x4, x3 x4])/v2}]
Out[7]= {x1^2 - x1 x2 + x2^2 + x1 x3 + x2 x3 + x3^2 + x1 x4 + x2 x4 + 2 x3 x4 + x4^2,
         -3 (x3 + x4)}
In[8]:= ω = u1 v1 + u2 v2 // Expand
Out[8]= x1^3 + x2^3 - x3^3 - x4^3
In[9]:= Simplify[Q.Q == ω IdentityMatrix[4]]
Out[9]= True
```

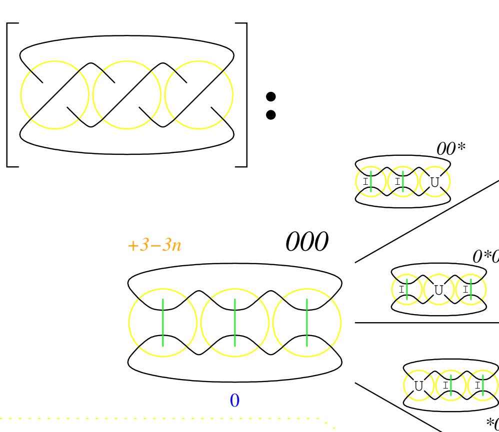
Example: Set $P=d$

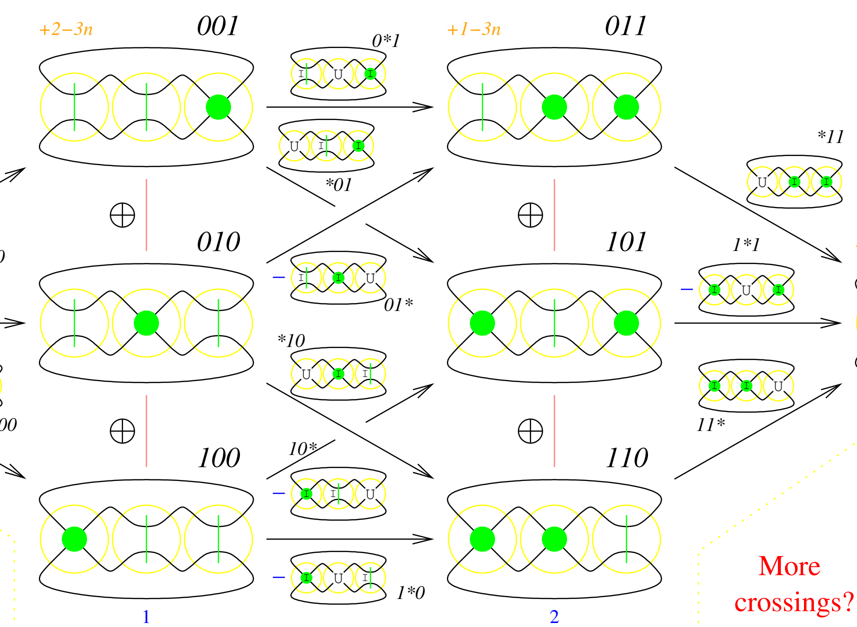



```
In[10]:=
P = ( 0        0        x1 - x4   x2 - x3 )
    ( 0        0        π[2,3]    -π[1,4] )
    ( π[1,4]   x2 - x3  0         0       )
    ( π[2,3]   x4 - x1  0         0       );
In[11]:=
Simplify[P.P == ω IdentityMatrix[4]]
Out[11]=
True
In[12]:=
U = ( x4 - x2                              0  0   0   )
    ( (u1 + x4 u2 - π[2,3])/(x1 - x4)      1  0   0   )
    ( 0                                    0  x4  -x2 )
    ( 0                                    0  -1  1   );
V = ( 1                                    0        0  0  )
    ( (u1 + x1 u2 - π[2,3])/(x4 - x1)      x1 - x3  0  0  )
    ( 0                                    0        1  x3 )
    ( 0                                    0        1  x1 );
Simplify[{U.P == Q.U, V.Q == P.V}]
Out[12]=
{True, True}
```

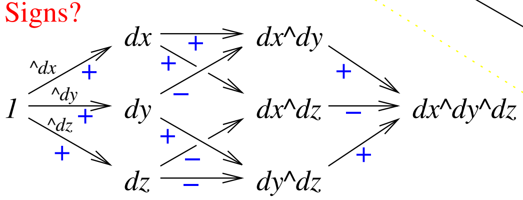
Crossings.







Signs?



More crossings?