FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
EXERCISES HANDOUT # 9

1. Show that S^3 is diffeomorphic with $SU(2)$, so that S^3 is a Lie group. Let $\mathcal{G} = T_e S^3$. Find a presentation of $\mathcal{G}$ as a Lie algebra.

2. Show that there is a one to one correspondence between
(a) Vector fields on M.
(b) Derivations $C^\infty(M) \to C^\infty(M)$.

3. Let M be a smooth manifold. A flow on M is an action (in the category of smooth manifolds) of the group $\mathbb{R}$ on M.

Show that if M is a compact manifold, then there is a one to one correspondence between flows and vector fields.

4. Let M be a smooth manifold and X, Y be vector fields that in a small trivial open set U are given by

$$\begin{aligned} X &= \Sigma_{i=1}^n f_i \frac{\partial}{\partial x_i} \\ Y &= \Sigma_{i=1}^n g_i \frac{\partial}{\partial x_i}. \end{aligned}$$

(a) Compute $[X, Y]$ in this local coordinate system.
(b) Prove the following "geometric" interpretation of $[X, Y]$ as the difference between flowing first along Y and then along X and flowing first along X and then along Y.

Suppose we work in the same trivial neighborhood U of a point p. So $U \equiv \mathbb{R}^n$ and $T_u U$ is identified naturally with $\mathbb{R}^n$. For a vector field Z define $J_{Z,\epsilon} : U \to U$ by

$$J_{Z,\epsilon} : x \mapsto x + \epsilon Z(x).$$

Show that

$$[X, Y](p) = \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} [J_{X,\epsilon} J_{Y,\epsilon} - J_{Y,\epsilon} J_{X,\epsilon}](p)$$

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