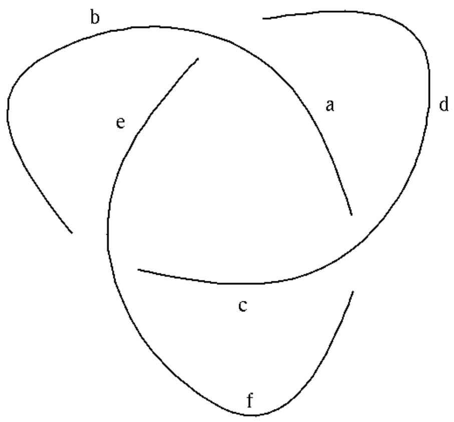
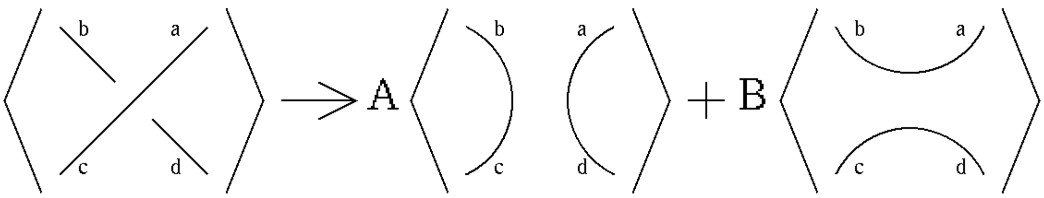


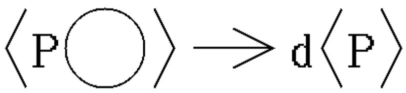

```
In[1]:= Trefoil = X[a, d, b, e] X[e, b, f, c] X[c, f, d, a]

Out[1]= X[a, d, b, e] X[c, f, d, a] X[e, b, f, c]
```

The definition of the Kauffman Bracket is given by the skein relations



and



The first rule can be coded as follows :

```
In[2]:= rule1 = (X[a_, b_, c_, d_] :> A δ[a d] δ[b c] + B δ[a b] δ[c d]);
```

Let as apply the first rule to the trefoil knot, just as an example:

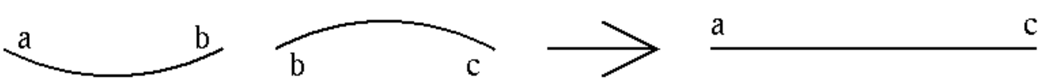
```
In[3]:= Trefoil /. rule1

Out[3]= (A δ[b d] δ[a e] + B δ[a d] δ[b e])
         (A δ[c e] δ[b f] + B δ[b e] δ[c f]) (B δ[a d] δ[c f] + A δ[a c] δ[d f])
```

```
In[4]:= Trefoil /. rule1 // Expand

Out[4]= A^2 B δ[a d] δ[b d] δ[a e] δ[c e] δ[b f] δ[c f] + A B^2 δ[a d]^2 δ[b e] δ[c e] δ[b f] δ[c f] +
         A B^2 δ[a d] δ[b d] δ[a e] δ[b e] δ[c f]^2 + B^3 δ[a d]^2 δ[b e]^2 δ[c f]^2 +
         A^3 δ[a c] δ[b d] δ[a e] δ[c e] δ[b f] δ[d f] + A^2 B δ[a c] δ[a d] δ[b e] δ[c e] δ[b f] δ[d f] +
         A^2 B δ[a c] δ[b d] δ[a e] δ[b e] δ[c f] δ[d f] + A B^2 δ[a c] δ[a d] δ[b e]^2 δ[c f] δ[d f]
```

We see that we need a rule that will dispose of unnecessary intermidiate points :




```
In[5]:= rule2 = (δ[a_ b_] δ[b_ c_] :> δ[a c]);
```

```
In[6]:= (Trefoil /. rule1 // Expand) //. rule2
```

Out[6]= $A^2 B \delta[c f]^2 + A B^2 \delta[a d]^2 \delta[c f]^2 + A B^2 \delta[b e]^2 \delta[c f]^2 + B^3 \delta[a d]^2 \delta[b e]^2 \delta[c f]^2 + 2 A^2 B \delta[d f]^2 + A B^2 \delta[b e]^2 \delta[d f]^2 + A^3 \delta[c e]^2 \delta[d f]^2$

This done, we can code the second rule, (using dd instead of d, to avoid naming conflicts)

```
In[7]:= rule3 = {(δ[_])^2 -> dd, δ[_^2] → dd};
```

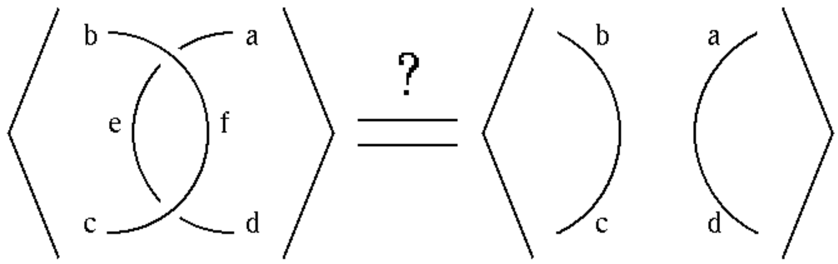
and see what heppens to the trefoil knot:

```
In[8]:= RawBracket[t_] := Simplify[(t /. rule1 // Expand) //. rule2 /. rule3]
```

```
In[9]:= RawBracket[Trefoil]
```

Out[9]= $dd\,(3 A^2 B + A^3 dd + 3 A B^2 dd + B^3 dd^2)$

Let's verify the second Reidemeister move :




```
In[10]:= RawBracket[X[b, e, f, a] X[f, e, c, d]] == δ[a d] δ[b c]
```

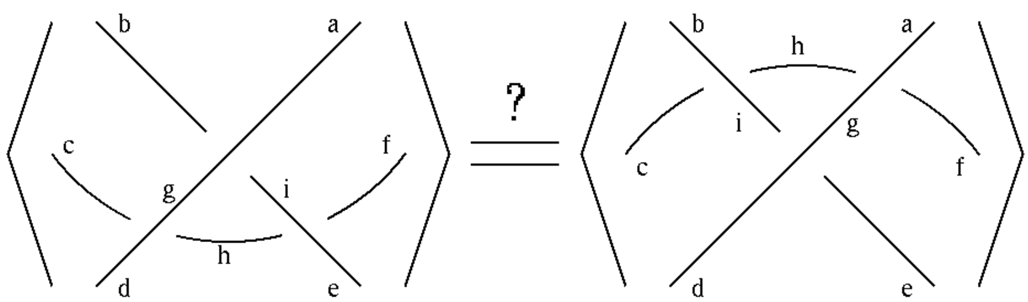
Out[10]= $A B \delta[b c] \delta[a d] + (A^2 + B^2 + A B\, dd) \delta[a b] \delta[c d] == \delta[b c] \delta[a d]$

Thus we see that for the second Reidemeister move to hold, we must have $A B == 1$ and $A^2 + B^2 + A B d == 0$. Solving for B and dd in terms of A we get:

```
In[11]:= rule4 = {B -> 1/A, dd -> -A^2 - 1/A^2}
```

Out[11]= $\{B \to \frac{1}{A},\ dd \to -\frac{1}{A^2} - A^2\}$

Time to check the third Reidemeister move : (It's guaranteed to hold, but loose nothing by checking again)




```
In[12]:= Bracket[t_] := Simplify[RawBracket[t] / dd /. rule4]
```

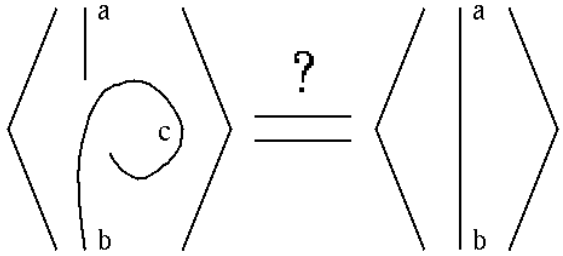
```
In[13]:= Bracket[X[a, b, g, i] X[g, c, d, h] X[i, h, e, f]]
```

Out[13]= $-\frac{1}{1+A^4}(A\,(\delta[a\,b]\,\delta[d\,e]\,\delta[c\,f]+\delta[c\,d]\,(A^4\,\delta[b\,e]\,\delta[a\,f]+A^2\,\delta[a\,b]\,\delta[e\,f])+\delta[b\,c]\,(A^2\,\delta[d\,e]\,\delta[a\,f]+\delta[a\,d]\,\delta[e\,f])))$

```
In[14]:= Bracket[X[b, c, i, h] X[d, e, g, i] X[g, f, a, h]]
```

Out[14]= $-\frac{1}{1+A^4}(A\,(\delta[a\,b]\,\delta[d\,e]\,\delta[c\,f]+\delta[c\,d]\,(A^4\,\delta[b\,e]\,\delta[a\,f]+A^2\,\delta[a\,b]\,\delta[e\,f])+\delta[b\,c]\,(A^2\,\delta[d\,e]\,\delta[a\,f]+\delta[a\,d]\,\delta[e\,f])))$

The remaining sad point is the behavior of the bracket under the first Reidemeister move :




```
In[15]:= Simplify[RawBracket[X[c, a, b, c]] /. rule4]
```

Out[15]= $-A^3\,\delta[a\,b]$

This is unfortunate, but not fatal. It can be fixed by multiplying the bracket by $-A^3$ raised to an appropriate multiple of the writhe is common to make the further substiturion $A \to q^{1/4}$ and then the resulting invariant is called the Jones polynomial. Let us

```
In[16]:= Jones[t_, w_] := Simplify[(Bracket[t] (-A^3)^(-w)) /. A → q^(1/4)]

In[17]:= Jones[Trefoil, 3]
```

Out[17]= $\frac{-1+q+q^3}{q^4}$

```
In[18]:= MirrorTrefoil = Trefoil /. x_X :→ RotateLeft[x]

Out[18]= X[b, f, c, e] X[d, b, e, a] X[f, d, a, c]

In[19]:= Jones[MirrorTrefoil, -3]
```

Out[19]= $q+q^3-q^4$

Ok, if you insist,

```
In[20]:= w[K_] := Module[
           {p, l},
           l = ReplacePart[K /. X[a_, b_, c_, d_] :→ δ[a c] δ[b d], p, {1, 0}] //.
             {p[a_ b_] :→ p[a, b], p[a_, b___, c_] δ[c_ d_] :→ p[a, b, c, d]};
           l = MapThread[Rule, {List @@ l, Range[Length[l]]}];
           (Plus @@ (K /. l)) /.
            X[a_, b_, c_, d_] :→ If[Abs[p = (a - c) (b - d)] == 1, p, -Sign[p]]
          ]

In[21]:= Jones[K_] := Jones[K, w[K]]
```

```
In[22]:= {Jones[Trefoil], Jones[MirrorTrefoil]}
```

Out[22]= $\left\{\frac{-1+q+q^3}{q^4},\ q+q^3-q^4\right\}$

Now, last but not least,

```
In[23]:= Expand[(
            q * (-A^3) ^ (-1) * RawBracket[X[a, b, c, d]] -
             q^ (-1) * (-A^3) * RawBracket[X[b, c, d, a]]
           ) /. {A → q^ (1 / 4), B → q^ (-1 / 4)}
         ]
```

Out[23]= $\frac{\delta[b\ c]\ \delta[a\ d]}{\sqrt{q}} - \sqrt{q}\ \delta[b\ c]\ \delta[a\ d]$