ALGEBRAIC STRUCTURES ON KNOTTED OBJECTS AND UNIVERSAL FINITE TYPE INVARIANTS

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Abstract. We discuss a number of topics related to algebraic constructions of universal finite type invariants. The idea is to find presentations of knot theory, or of some mild generalizations of knot theory, in terms of finitely many generators and relations, and then to construct a universal finite type invariant by setting its values on the generators so as the relations are satisfied. One such presentation involves knotted trivalent graphs, and is genuinely 3-dimensional. In this presentation the generators turns out to be the standardly embedded tetrahedron ◬ and the relations are on one hand equivalent to the pentagon and hexagon relations of Drinfel'd's theory of associators and on the other hand they are closely related to the Biedenharn-Elliot identities of $6j$-symbols and to the Pachner moves of the theory of triangulations. Another such presentation involves Jones' notion of a planar algebra [J] and leads to a crossing-centric constructions of a universal finite type invariant (as opposed to the now-standard associativity-centric construction). Much of what we discuss is work in progress, and this article contains many "live ends", unfinished problems that don't seem to be dead ends.

This work in progress is available electronically at `http://www.ma.huji.ac.il/~drorbn/Misc`.

Contents

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1. Introduction

1.1. Finite type invariants and the fundamental theorem.

Summary. A brief introduction to the fundamental theorem and universal finite type invariants. You can find all that (and more) in [BNS].

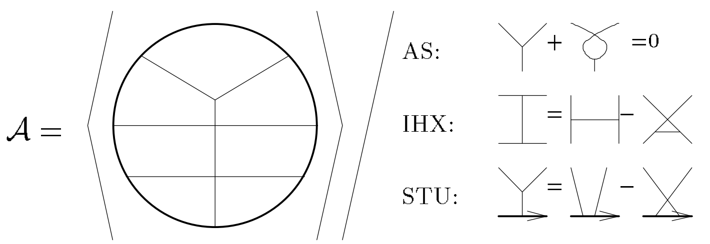
The *Fundamental Theorem of Finite Type Invariants*[1] has a deceptively simple formulation and a surprising number of proofs and partial proofs, each one coming with its own philosophy and employing its own set of tools. The purpose of this article is to further study one family of approaches, the *algebraic approaches*, not so much as to prove the theorem, for this summit is already multiply climbed, but rather for the mere beauty of these specific paths, for the view from some of the vista points along those, for some new perspectives and insights gained along the way, and, well, o.k., also for an occassional technical advantage over the other approaches. In fact, some of the paths we will take don't even make it all the way to the top, or if they do, they are sometimes obviously non-geodetic, but to find the nearby shorter routes one would have to venture a bit into still unexplored territory. So another reason for the existance of this article is to encourage ourselves, and others, to complete and improve what we already have. Thus quoting from the abstract, this article suggests many "live ends", unfinished problems that don't seem to be dead ends.

In one form, the Fundamental Theorem of Finite Type Invariants, or just the *Fundamental Theorem* throughout this article, asserts that there exists a universal finite type invariant, an essential invariant $Z : \mathcal{K}(\circlearrowleft) \to \mathcal{A}(\circlearrowleft)$. Let us start by defining the terms in this statement, and then, in the further parts of this introduction, we will sketch the algebraic family of approaches (to be followed by full details in Sections 2–4):

[1]Also known as the *Fundamental Theorem of Vassiliev Invariants* or *Kontsevich's Theorem* or simply *Kontsevich's Integral* after the main ingredient of its first proof.

- $\mathcal{K}(\circlearrowleft)$ is the set of all framed knots. When we draw knots in the plane[2] we will always assume the standard blackboard framing, except when using the notations ⤳ and ⤴, which indicate a right-handed and left-handed framing twists, respectively. Thus for example,

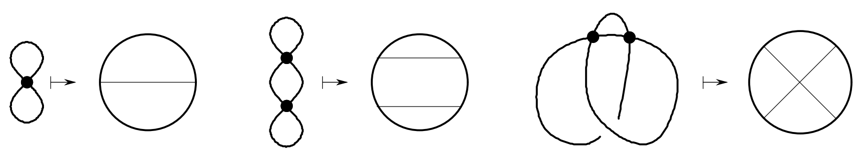
- $\mathcal{A}(\circlearrowleft)$ is the usual graded-completed algebra of chord diagrams whose skeleton is a circle, modulo AS, IHX and STU relations[3]:



- We now come to the only condition that Z has to satisfy, that it be "essential". Recall first that any knot invariant with values in an Abelian group can be extended to n-*singular knots*, knots with n double point, by iterated use of the local formula

$$Z(\times) = Z(\times) - Z(\times).$$

Recall also that the *symbol* D_K of an n-singular knot K is the degree n chord diagram obtained by taking the parameter space of K, a circle, and connecting by a chord any pair of points on that circle that are identified in the image:

(1) $K \mapsto D_K$:  .

The "essential" condition on Z is that whenever K is an n-singular knot,

(2) $$Z(K) = D_K + \text{terms of higher degree.}$$

An equivalent formulation of the Fundamental Theorem is that every degree n weight system is the nth derivative of some type n invariant; in particular, it follows that there are lots of finite type invariants, and it reduces the problem of their enumeration to a finite algebro-combinatorial problem at any fixed degree.

1.2. Generators, relations and syzygies.

Summary. As a toy model for the algebraic approach to the construction of Z, we give a brief introduction to generators, relations and syzygies in a group-theoretical context, and their use in the construction of group representations.

As we have already mentioned, there are many approaches to the construction of an invariant $Z : \mathcal{K}(\circlearrowleft) \to \mathcal{A}(\circlearrowleft)$ satisfying the condition in Equation (2). The algebraic approach, which is the topic of this article, is to find some algebraic context within which the set $\mathcal{K}(\circlearrowleft)$ (or some mild generalization thereof) is finitely presented, and then to use this finite

[2] As we are often forced to do due to the limitations of our media.

[3] If these terms don't seem terribly "usual" to you, you should fix that before proceeding. Being totally unbiased, our favorite reference is [BN1].

presentation to define Z. Namely, one would have to make wise guesses $Z(K_i)$ for the values of Z on the generators K_i of $\mathcal{K}(\circlearrowleft)$, so that for each relation $R_j(K_1,\ldots)$ the corresponding values of Z would satisfy the corresponding relation $R_j(Z(K_1),\ldots)$ (two comments: 1. For this to make sense $\mathcal{A}(\circlearrowleft)$ must carry the same kind of algebraic structure as $\mathcal{K}(\circlearrowleft)$; 2. The verification of essentiality, Equation (2), is typically easy).

Let us see what this entails on a toy model. Suppose we want to find invariants of elements of the set B_4 of braids on 4 strands. One way to proceed is to notice that B_4 carries an algebraic structure, that is, it has an associative product which makes it a group. Thus we may seek invariants on B_4 with values in associative algebras, which respect the algebraic structure. Such creatures are not new on the mathematical scenery; they are usually called "group representations". Our approach to finding representations of B_4 would be to make wise guesses for their values $Z(\sigma_1)$, $Z(\sigma_2)$ and $Z(\sigma_3)$ on the generators σ_1, σ_2 and σ_3 of B_4 (see Figure 1), so as to satisfy the relations between the σ_i's. Setting $z_i := Z(\sigma_i)$, these relations are (again see Figure 1):

$$z_1z_3 = z_3z_1, \quad z_1z_2z_1 = z_2z_1z_2 \quad \text{and} \quad z_2z_3z_2 = z_3z_2z_3. \tag{3}$$

In our real problem, the construction of $Z : \mathcal{K}(\circlearrowleft) \to \mathcal{A}(\circlearrowleft)$, the target space $\mathcal{A}(\circlearrowleft)$ is graded, and we will attempt to construct Z inductively, degree by degree. Thus we will be asking ourselves, "suppose our construction is done to degree 16; can we extend it to degree 17?". Let us go back to the toy model and examine the situation over there. Let A be an associative algebra and let $J \subset I \subset A$ be ideals in A (think "$I = \{\text{degrees} \geq 17\}$ and $J = \{\text{degrees} > 17\}$") so that $I \cdot I \subset J$ ("$17 + 17 > 17$"). Suppose we have $z_i \in A$ which satisfy the equations (3) in A/I ("done to degree 16"). But equations (3) may fail in A/J; let $\lambda, \psi_a, \psi_b \in I/J$ be the errors in when these equations are considered in A/J:

$$\lambda := z_1z_3 - z_3z_1, \quad \psi_a := z_1z_2z_1 - z_2z_1z_2 \quad \text{and} \quad \psi_b := z_2z_3z_2 - z_3z_2z_3. \tag{4}$$

We wish to modify the z_i's so as to satisfy equations (3) in A/J ("extend to degree 17"), so we set

$$z_i' = z_i + \zeta_i \in A/J, \tag{5}$$

where we assume that $\zeta_i \in I/J$ (so as $z_i = z_i'$ in A/I) ("the correction ζ_i is of degree precisely 17"). We now compute the new errors $\lambda', \psi_a', \psi_b' \in I/J$ in terms of the old ones and using the property $I \cdot I \subset J$:

$$\begin{aligned} \lambda' &:= z_1'z_3' - z_3'z_1' = (z_1+\zeta_1)(z_3+\zeta_3) - (z_3+\zeta_3)(z_1+\zeta_1) \\ &= z_1z_3 - z_3z_1 + z_1\zeta_3 + \zeta_1z_3 - z_3\zeta_1 - \zeta_3z_1 \\ &= \lambda + \zeta_1z_3 + z_1\zeta_3 - \zeta_3z_1 - z_3\zeta_1, \end{aligned} \tag{6}$$

and likewise,

$$\begin{aligned} \psi_a' &= \psi_a + \zeta_1z_2z_1 + z_1\zeta_2z_1 + z_1z_2\zeta_1 - \zeta_2z_1z_2 - z_2\zeta_1z_2 - z_2z_1\zeta_2, \\ \psi_b' &= \psi_b + \zeta_2z_3z_2 + z_2\zeta_3z_2 + z_2z_3\zeta_2 - \zeta_3z_2z_3 - z_3\zeta_2z_3 - z_3z_2\zeta_3. \end{aligned}$$

These are linear equations, and thus to solve our problem, namely to find ζ_i's so that $\lambda' = \psi_a' = \psi_b' = 0$, we need to show that the triple $E := (\lambda, \psi_a, \psi_b) \in A^3$ is in the image of

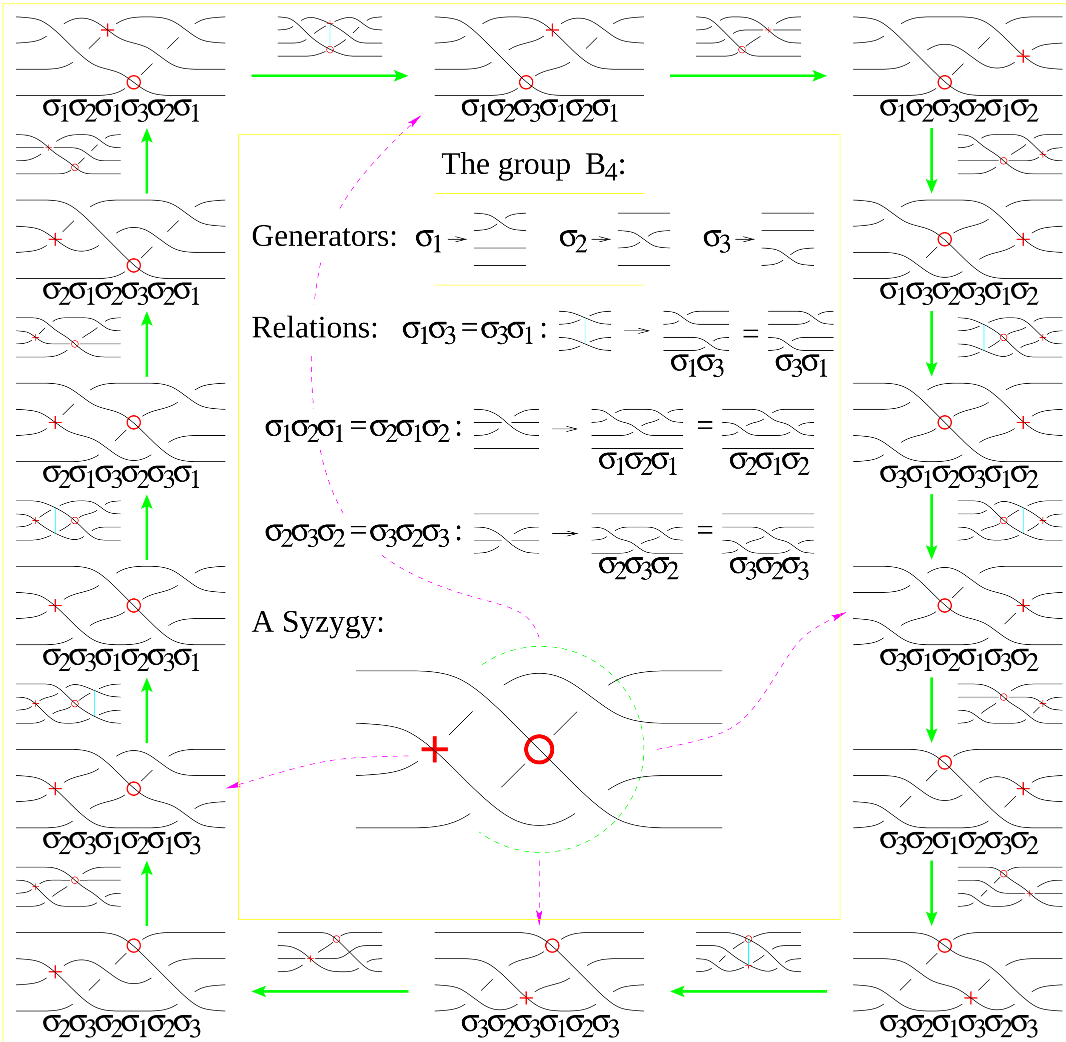


Figure 1. A finite presentation of the group B_4 and one of its syzygies. The central frame in this figure shows the generators σ_1, σ_2 and σ_3 of B_4 followed by a standard way of writing the relations between them. The outside frame shows a syzygy between these relations — a closed loop whose vertices are words in the generators σ_i and whose edges are relations. See Comment 6.1.

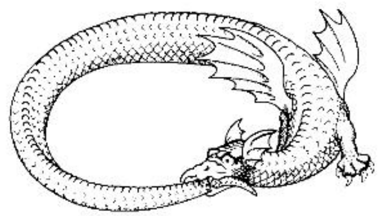
the linear map $d^r : A^3 \to A^3$ defined by

$$d^r : \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix} \mapsto \begin{pmatrix} \zeta_1 z_3 + z_1\zeta_3 - \zeta_3 z_1 - z_3\zeta_1 \\ \zeta_1 z_2 z_1 + z_1\zeta_2 z_1 + z_1 z_2\zeta_1 - \zeta_2 z_1 z_2 - z_2\zeta_1 z_2 - z_2 z_1\zeta_2 \\ \zeta_2 z_3 z_2 + z_2\zeta_3 z_2 + z_2 z_3\zeta_2 - \zeta_3 z_2 z_3 - z_3\zeta_2 z_3 - z_3 z_2\zeta_3 \end{pmatrix}.$$

Our strategy to show that $E \in \operatorname{im} d^r$ is to find a second linear map d^s, whose domain is the target space of d^r, so that $d^s \circ d^r = 0$ and so that $d^s E = 0$. This done we can define the homology group $H := \ker d^s / \operatorname{im} d^r$, and if by some magical means we could prove that it vanishes, we would use $d^s E = 0$ to determine that $E \in \operatorname{im} d^r$, and our problem would be solved. We will mention techniques for the computation of the homology group H in Section 5. For now we only wish to describe how the map d^s is found.

To find linear relations between the errors λ, ψ_a and ψ_b, we start with a *syzygy* for our presentation of the braid group B_4 — a closed loop whose vertices are words in the generators σ_i and whose edges are relations. When we perform the replacement $\sigma_i \to z_i$ on the vertices of a syzygy, say the one displayed in Figure 1, we get a loop like such:

(7)
$$\begin{array}{ccccc}
z_1z_2z_1z_3z_2z_1 & \xrightarrow[-z_1z_2\lambda z_2z_1]{} & z_1z_2z_3z_1z_2z_1 & \xrightarrow[-z_1z_2z_3\psi_a]{} & z_1z_2z_3z_2z_1z_2 \\
\uparrow{\scriptstyle \psi_az_3z_2z_1} & & & & {\scriptstyle -z_1\psi_bz_1z_2}\downarrow \\
z_2z_1z_2z_3z_2z_1 & & & & z_1z_3z_2z_3z_1z_2 \\
\uparrow{\scriptstyle z_2z_1\psi_bz_1} & & & & {\scriptstyle -\lambda z_2z_3z_1z_2}\downarrow \\
z_2z_1z_3z_2z_3z_1 & & & & z_3z_1z_2z_3z_1z_2 \\
\uparrow{\scriptstyle z_2\lambda z_2z_3z_1} & & & & {\scriptstyle z_3z_1z_2\lambda z_2}\downarrow \\
z_2z_3z_1z_2z_3z_1 & & & & z_3z_1z_2z_1z_3z_2 \\
\uparrow{\scriptstyle -z_2z_3z_1z_2\lambda} & & & & {\scriptstyle -z_3\psi_az_3z_2}\downarrow \\
z_2z_3z_1z_2z_1z_3 & & & & z_3z_2z_1z_2z_3z_2 \\
\uparrow{\scriptstyle z_2z_3\psi_az_3} & & & & {\scriptstyle -z_3z_2z_1\psi_b}\downarrow \\
z_2z_3z_2z_1z_2z_3 & \xleftarrow{\psi_bz_1z_2z_3} & z_3z_2z_3z_1z_2z_3 & \xleftarrow{-z_3z_2\lambda z_2z_3} & z_3z_2z_1z_3z_2z_3
\end{array}$$



Now given that the edges of a syzygy are relations, we know that the difference between the element written at the head of any given edge and at the tail of that edge is a multiple of λ, ψ_a or ψ_b. These multiples are written to the side of each edge in Equation (7). By the ouroboros[4] summation formula (a cousin of the telescopic summation formula, but where the beginning point and the end point are the same) the sum of these differences is 0. That is, E is in the kernel of the linear map d^s defined by

$$d^s : E = \begin{pmatrix} \lambda \\ \psi_a \\ \psi_b \end{pmatrix} \mapsto \begin{array}{l} -z_1z_2\lambda z_2z_1 - z_1z_2z_3\psi_a - z_1\psi_bz_1z_2 - \lambda z_2z_3z_1z_2 + z_3z_1z_2\lambda z_2 \\ -z_3\psi_az_3z_2 - z_3z_2z_1\psi_b - z_3z_2\lambda z_2z_3 + \psi_bz_1z_2z_3 + z_2z_3\psi_az_3 \\ -z_2z_3z_1z_2\lambda + z_2\lambda z_2z_3z_1 + z_2z_1\psi_bz_1 + \psi_az_3z_2z_1. \end{array}$$

Moral. It would be nice to have an algebraic context within which knot theory is finitely presented and within which the syzygies of the presentation are simple to analyze.

Problem 1.1. In the specific case of the presentation of Figure 1 of the braid group B_4 (and its obvious generalization to B_n), we don't know if the methodology of this section can actually be used to construct invariants (though we do know of some more complecated situations in which this methodology is useful; see the rest of this article). This is especially interesting when the target algebra A is taken to be the algebra $\mathcal{A}_n^b$ of chord diagrams for braids (see e.g. [BN4]).

1.3. Parenthesized tangles.

Summary. As an example where the scheme of Section 1.2 has been successfully used, and also in order to display some formulas for later use in this article, we give a very quick reminder of parenthesized tangles and the pentagon and hexagon relations and their syzygies, along the lines of [BN3, BN5, LM].

The papers [BN3, BN5, LM] introduce an algebraic context within which the scheme of Section 1.2 is used to construct a universal finite type invariants of links. The "algebraic

[4] http://www.draconian.com/whatis/whatis.htm

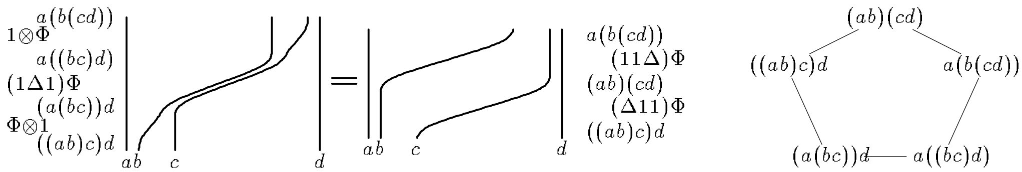


Figure 2. The pentagon relation ⬠ and its tensor-category-theoretical origin.

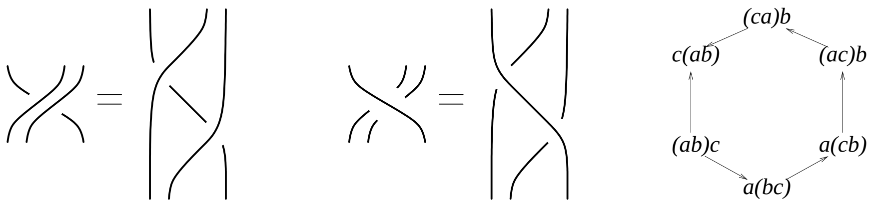
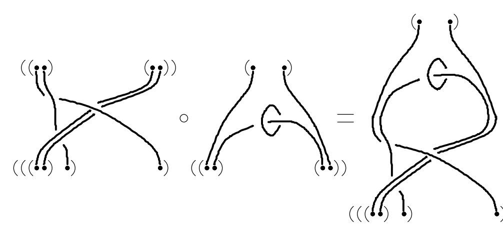
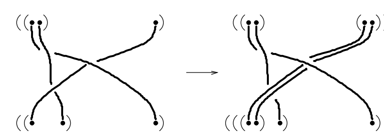


Figure 3. The positive and negative hexagon relations $⬡_\pm$ and their tensor-categorical origin.

context" there is the structure of a category with certain additional operations. Rather than defining everything in full, we will just recall some key notions, pictures and formulas here.



The category **PaT** of "parenthesized tangles", (the algebraic structure which we wish to represent, like B_4 in Section 1.2) is the category whose objects are *parenthesizations* such as $(((\cdot\cdot)\cdot)\cdot)$ or $((\cdot\cdot)(\cdot\cdot))$, and whose morphisms are tangles with parenthesized top and bottom. See the picture on the right, which also illustrates how parenthesized tangles are composed.



The category **PaT** carries some additional operations. The most interesting are the "strand addition on the left/right" operations, and the strand doubling operations (illustrated on the right). More details are in [BN3, BN5, LM].

Likewise, one can set up a category $\mathbf{Pa}\mathcal{A}$ of "parenthesized chord diagrams", that captures the "symbols" of "singular" parenthesized tangles as in Equation (1). The category $\mathbf{Pa}\mathcal{A}$ supports the same additional operations as **PaT**, and one may wish to look for structure preserving functors $Z : \mathbf{PaT} \to \mathbf{Pa}\mathcal{A}$ which are "essential" in a sense similar to that of Equation (2). In [BN3], this is done following the same generators-relations-syzygies sequence as in Section 1.2:

1.3.1. *Generators.* The category **PaT** is generated by the morphisms ⋈ and ×. We set $\tilde{\Phi} = Z(⋈)$ and $\tilde{R} = Z(\times)$ and then reconsider these morphisms in $\mathbf{Pa}\mathcal{A}$ as elements $\Phi \in \mathcal{A}(\uparrow_3)$ and $R \in \mathcal{A}(\uparrow_2)$, where $\mathcal{A}(\uparrow_n)$ denotes the usual space of chord diagrams modulo AS, IHX and STU relations on a skeleton made of n vertical lines.

1.3.2. *Relations.* There are several relations beween ⋈ and × as elements of the algebraic structure $\mathbf{Pa}\mathcal{A}$, as listed in [BN3]. The most prominent of those are the pentagon ⬠ and the two hexagon relations $⬡_\pm$, displayed in Figures 2 and 3.

Now let us assume that we already found R and Φ so that the relations between them corresponding to ⬠ and ⬡$_\pm$ are satisfied up to degree 16 (say), and let μ and $\psi_\pm$ be the degree 17 errors in these equations (compare with Equation (4)). That is, modulo degrees 18 and up we have (notaion as in [BN3], compare with [BN3, Equations (10) and (11)]):

$$\begin{aligned}\mu &= \Phi^{123}\cdot(1\otimes\Delta 1)(\Phi)\cdot\Phi^{234} - (\Delta\otimes 1\otimes 1)(\Phi)\cdot(1\otimes 1\otimes\Delta)(\Phi),\\ \psi_\pm &= \Phi^{123}\cdot(R^{\pm 1})^{23}(\Phi^{-1})^{132}\cdot(R^{\pm 1})^{13}\cdot\Phi^{312} - (\Delta\otimes 1)(R^{\mp 1})\end{aligned}$$

Proceeding as in Equation (5) we set $\Phi' = \Phi + \varphi$ and $R' = R + r$ with φ and r of degree 17, and like in Equation (6) we get (compare with [BN3, Equations (12) and (13)]):

$$\begin{aligned}\mu' &= \mu + \varphi^{234} - (\Delta\otimes 1\otimes 1)(\varphi) + (1\otimes\Delta 1)(\varphi) - (1\otimes 1\otimes\Delta)(\varphi) + \varphi^{123}\\ \psi'_\pm &= \psi_\pm + \varphi^{123} - \varphi^{132} + \varphi^{312} \pm \left(r^{23} - (\Delta\otimes 1)(r) + r^{13}\right).\end{aligned}$$

Thus we are interested in knowing whether the triple $E := (\mu, \psi_\pm)$ is in the image of the linear map

$$d^r : \begin{pmatrix}\varphi\\ r\end{pmatrix} \mapsto \begin{pmatrix}\varphi^{234} - (\Delta\otimes 1\otimes 1)(\varphi) + (1\otimes\Delta 1)(\varphi) - (1\otimes 1\otimes\Delta)(\varphi) + \varphi^{123}\\ \varphi^{123} - \varphi^{132} + \varphi^{312} \pm (r^{23} - (\Delta\otimes 1)(r) + r^{13})\end{pmatrix}.$$

1.3.3. *Syzygies.* On like in Section 1.2, the trick is to use syzygies between the ⬠ and ⬡$_\pm$ relations to reduce the problem to the computation of a homology group $H := \ker d^s / \operatorname{im} d^r$ where d^s is some other linear map, for which $d^s \circ d^r = 0$ and $d^s E = 0$. Again, this was carried out in full in [BN3]. Here we only reproduce the four syzygies we need to use (see Figure 4) and the resulting map d^s (the four components of d^s correspond to the syzygies in Figure 4 in the order $\boxed{\begin{smallmatrix}1&2\\3&4\end{smallmatrix}}$; the symbol $(\Delta 111)$ denotes $(\Delta\otimes 1\otimes 1\otimes 1)$ etc.; compare with [BN3, Equations (15–19)]):

$$d^s : \begin{pmatrix}\mu\\ \psi_\pm\end{pmatrix} \mapsto \begin{pmatrix}\mu^{2345} - (\Delta 111)(\mu) + (1\Delta 11)(\mu) - (11\Delta 1)(\mu) + (111\Delta)(\mu) - \mu^{1234}\\ \mu^{1234} - \mu^{1243} + \mu^{1423} - \mu^{4123} - \psi_+^{234} + (\Delta 11)(\psi_+) - (1\Delta 1)(\psi_+) + \psi_+^{124}\\ \mu^{1234} - \mu^{1324} + \mu^{3124} - \mu^{3142} + \mu^{3412} + \mu^{1342}\\ \qquad -\psi_+^{124} + (11\Delta)(\psi_+) - \psi_+^{123} - \psi_-^{342} + (\Delta 11)(\psi_-^{231}) - \psi_-^{341}\\ \psi_+^{213} - \psi_+^{123} + \psi_-^{231} - \psi_-^{321}\end{pmatrix}.$$

1.4. Plan of the paper.

TBW.

1.5. Acknowledgement.

We wish to thank Greg Kuperberg for comments and suggestions.

2. Knotted trivalent graphs

2.1. Knotted trivalent graphs and some moves between them.

Summary. Here we will define the class of knotted objects we will be working with (knotted trivalent graphs, **KTG**) and the elementary moves between them. For now, see [BN7, handout].

2.2. The set of knotted trivalent graphs is finitely generated.

Summary. Here we will show that with the elementary moves of the previous section, the set **KTG** is generated by just two of it's elements: the trivially embedded positively twisted Möbius band ⟳ and the trivially embedded tetrahedron △. For now, see [BN6, slides 11–16] and [BN7, handout].

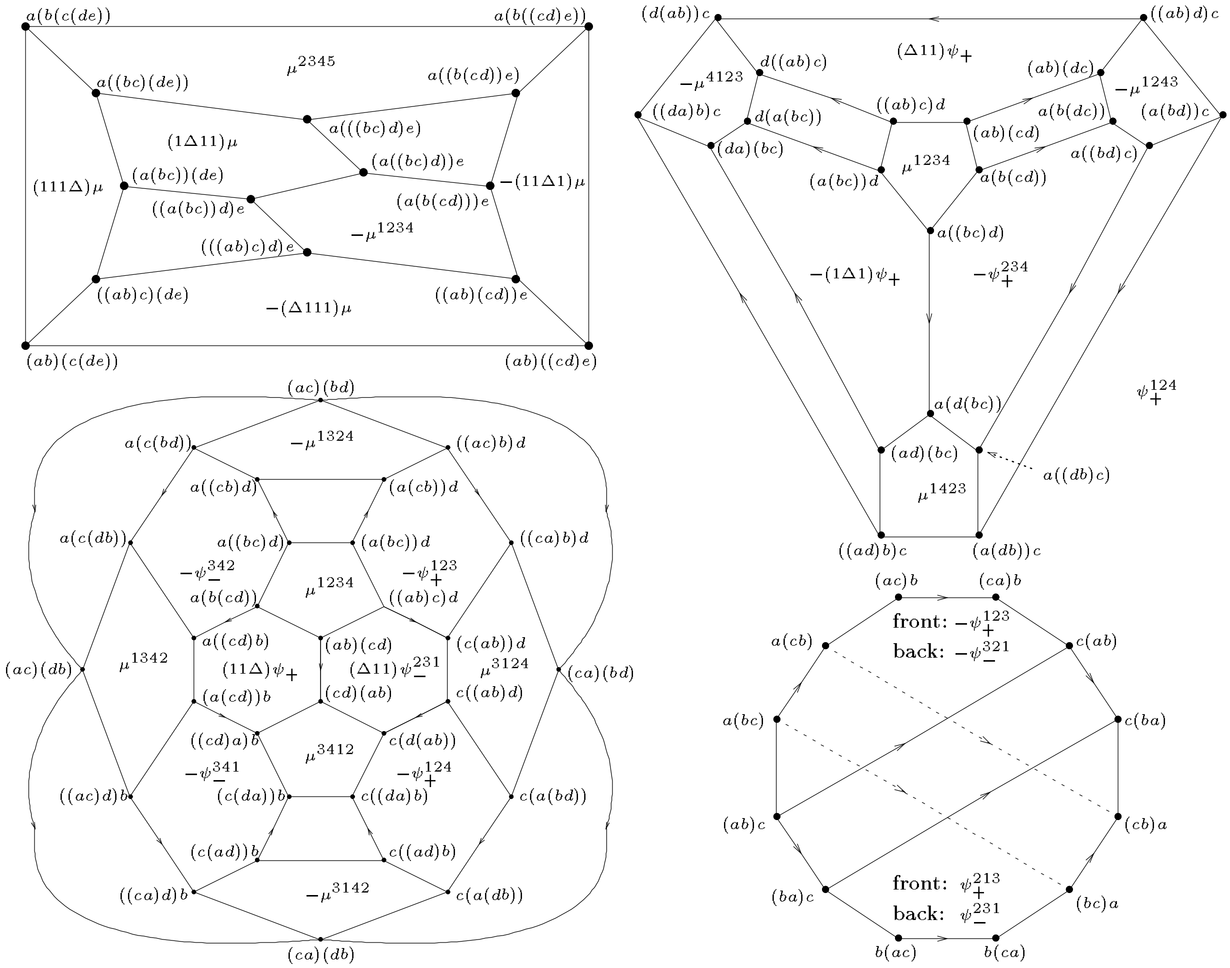


Figure 4. Four syzygies in **PaT**. The vertices in these pictures correspond to objects of **PaT**, the edges to morphisms, the faces to relations and hence each of the four polyhedra is a single syzygy. Further details are in [BN3, BN5].

2.3. Chord diagrams on trivalent skeleta.

Summary. This is the relevant space of chord diagrams for invariants of knotted trivalent graphs. For now, see [MO].

2.4. Symmetries, the pentagon and the hexagon relations.

Summary. Here we discuss some relations between the generators ↻ and ⩓ of **KTG** and show that they are equivalent to Drinfel'd's pentagon and hexagon relations [Dr1, Dr2]. For now, see [BN7, handout].

2.5. Vertex renormalizations and uniqueness.

Summary. Here we will talk about the uniqueness up to vertex renormalizations of a well behaved universal invariant of knotted trivalent. This is the parallel in our theory of the uniqueness up to gauge equivalence of well behaved invariants of q-tangles [LM].

2.6. Is KTG finitely presented?

Summary. This is a sticky point. We are quite sure that relative to the elementary moves and with the generators $\circlearrowleft$ and $\triangle$ the set **KTG** is finitely presented, and we are quite sure that we know all the relations, and they are the pentagon and hexagon of Section 2.4. But depending on our mood in the morning of any given day, we either don't have a proof or are very unhappy about the proof we have. No reference yet.

2.7. The relation with $6j$-symbols and Turaev-Viro invariants.

Summary. Here we will explain how within our context $Z(\triangle)$ is related to the theory of $6j$-symbols. A nice Lie algebra problem still remains. No reference yet.

2.8. The relation with perturbative Chern-Simons theory.

Summary. Our discussion so far implies that if one could set up a well behaved perturbative Chrn-Simons theory (synonymously, a well behaved theory of configuration space integrals), then the invariant of the tetrahedron $\triangle$ would be an associator, when viewed in the right way. We plan a short discussion of this matter here. No reference yet.

2.9. Some dreams regarding Witten's Asymptotics Conjecture and the Kashaev-Murakami-Murakami Volume Conjecture.

Summary. We have some very speculative remarks (that in fact where the origin of this whole study) as for the relationship between everything here and the Witten's Asymptotics Conjecture (that the asymptotics of the Reshetikhin-Turaev invariants is governed by Feynmann-diagram expansions around flat connections) and its sibling the Kashaev-Murakami-Murakami Volume Conjecture. Little as we have to say about it, we'll say it here. For now, see [BN6].

3. Planar Algebras

Summary. Another approach for understanding knot theory as a finitely presented theory is within the context of planar algebras which we will review is Section 3.1. Within the context of planar algebras, knot theory has a very nice (and familiar) decription — it is the theory generated by crossings modulo the standard Reidemeister moves. Even the syzygies of this theory are simply enumerated by codimension two singular plane projections. For now, see [BN8, BN9].

3.1. A quick introduction to planar algebras. For now, see [J].

3.2. The standard planar algebras of tangles and of chord diagrams and the no-go theorem.

Summary. Here we will describe the standard planar algebras of tangles and its associated standard algebra of chord diagrams, and show that there is essential planar algebra map from the former planar algebra to the latter. For now, see [BN9, slide 16].

3.3. Shielded chord diagrams and tangles.

Summary. There is a "better" planar algebra of chord diagrams, that does support a universal finite type invariant. We will define it here. For now, see [BN9, slides 17–18].

3.4. Generators, relations and syzygies.

Summary. Here is where we will go through the generators-relations-syzygies sequence in the case of the planar algebra of tangles. Nothing's formally written yet.

4. Annular braids

Summary. This is another semi-successful algebraic approach to the construction of a universal finite type invariants via generators and relations. The central objects here are "annular braids", braids in an annulus cross an interval (rather than a disk cross an interval). Nothing written yet.

5. Appendix: Symmetrization and Anonymization

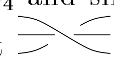
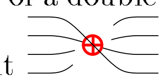
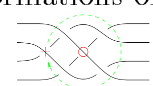
Summary. Here we will recall the three-step reduction of chord-diagram-valued equations-given-constraints problems into manageable homological algebras problems, as in [Dr2, LM, BN3].

5.1. **Linear substitution problems.** TBW.

5.2. **Symmetrization.** TBW.

5.3. **Anonymization.** TBW.

6. Appendix: Comments

6.1. A comment on Figure 1. It would be worthwhile for the reader to reflect on the relationship between the relations and the syzygies of B_4 and singularities of plane curves. One such codimension one singularity is the triple point , which corresponds to the last two relations above, which can be viewed as "the motion of a double point across a line". One such codimension two singularity is the quadruple point , and it corresponds to the syzygy of Figure 1: There is a circle-worth of generic deformations of the quadruple point, corresponding to "the cross rotating around the target": . The different codimension one singularities along this rotation are exactly the relations in our syzygy.

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